The Application of a Conservative Grid Adaptation Technique to 1D Shallow Water Equations

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Abstract—In this contribution, we present an efficient conservative mesh adaptation algorithm applied to 1D shallow water equations. This algorithm is suitable for unsteady situations and discontinuities of the solutions are well captured. Numerical results are presented. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Grid adaptivity, Q-schemes of Roe and Van Leer, Implicit and explicit TVD methods, Unsteady shallow water flows

1. INTRODUCTION

Numerical methods for predicting the water profile and discharge in steady as well as unsteady situations of hydraulic systems have become a common tool. In particular, finite difference applications of numerical schemes have been widely reported.

One of the basic problems in unsteady hydraulic systems is the location of solution discontinuities and shocks. In order to solve this problem, an efficient conservative grid adaptation algorithm applied to the resolution of the shallow water equations is presented.

First, the equations to be solved are presented. They are essentially the well-known shallow water equations written in conservative form. The discretisation of the system is done using the numerical method proposed by Bermúdez-Vázquez in [1]. A high-order method as the TVD-McCormack scheme (see [2,3]) has also been used to compare numerical solutions.

A posteriori error estimator to control the error of the numerical solution is constructed using a metric tensor $M$ ($M$ being the solution of a minimisation problem). Once $M$ is computed, a simple version of an anisotropic Delaunay algorithm for one-dimensional domains is used to adapt the mesh. A local conservative interpolation algorithm is used to guarantee conservation of the variables during mesh adaptation.

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Finally, numerical results are presented and comparisons with nonconservative interpolation algorithms and other numerical schemes are given.

2. SHALLOW WATER EQUATIONS.
   NUMERICAL DISCRETISATION

Shallow water equations represent mass and momentum conservation along the direction of the main flow. They constitute an adequate description for most of the problems associated with open channel flow modeling and can be written as the following system of equations:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0,$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} + gI_1 \right) = gI_2 + gA(S_0 - S_f),$$

where $A$ is the wetted cross-sectional area, $Q$ is the discharge, $g$ is the acceleration due to gravity, and $I_1$ represents a hydrostatic pressure force term

$$I_1 = \int_0^{h(x,t)} (h - \eta)B(x) \, d\eta,$$

and $I_2$ accounts for the pressure forces due to longitudinal width variations,

$$I_2 = \int_0^{h(x,t)} (h - \eta)B'(x) \, d\eta.$$

$B(x)$ is the breadth of the channel that is supposed to be locally rectangular. The right-hand side of equation (1) also contains the sources and sinks of momentum arising from the bed slope and friction losses.

The discretisation of the system is done using the numerical method proposed by Bermúdez-Vázquez in [1], that is, an explicit extension of the $Q$-scheme of Roe with upwinding in flux and source terms. For more details, see [1,4,5] A high-order method as the TVD-McCormack scheme (see [2,3]) has also been used to compare numerical solutions.

3. ERROR ESTIMATOR: METRIC COMPUTATION

A posteriori error estimator to control the error of the numerical solution is constructed using a metric tensor $\mathcal{M}$ ($\mathcal{M}$ being the solution of the following minimisation problem) (see [6,7] for more details) find a metric tensor $\mathcal{M}$, so that the adapted Delaunay mesh constructed from $\mathcal{M}$ minimises the interpolation error

$$\|W^n - \Pi_n W^n\|_{L^\infty(T_h)},$$

where $W^n = (A^n, Q^n)^T$ is the solution of the shallow water system at time $t = t_n$ and $\Pi_n W^n$ is a continuous piecewise linear interpolation of $W^n$ over the mesh $T_h$. In general, this metric tensor is given in terms of the Hessian matrices of the variables. In this particular case, as $A^n$ and $Q^n : \mathbb{R}^2 \rightarrow \mathbb{R}$, the metric tensor can be computed as

$$\mathcal{M}(x) = \max \left\{ T_{\epsilon_0}^{\epsilon_1} \left( \frac{|D^2 A^n(x)|}{e_0 A^n \|L^\infty(T_h)\|} \right), T_{\epsilon_0}^{\epsilon_1} \left( \frac{|D^2 Q^n(x)|}{e_0 Q^n \|L^\infty(T_h)\|} \right) \right\},$$

where $D^2 A^n$ and $D^2 Q^n$ are, respectively, the second derivatives of $A^n$ and $Q^n$, $e_0$ is a positive control parameter, and $T_{\epsilon_0}^{\epsilon_1}$ a truncated function that avoids metric degeneration

$$T_{\epsilon_0}^{\epsilon_1}(x) = \begin{cases} x, & \text{if } \epsilon_0 < x < \epsilon_1, \\ e_0, & \text{if } x \leq \epsilon_0, \\ \epsilon_1, & \text{if } x \geq \epsilon_1. \end{cases}$$
where, usually, $\epsilon_0 = 1/t_{\text{max}}^2$ and $\epsilon_1 = 1/t_{\text{min}}^2$, being $t_{\text{max}}$ and $t_{\text{min}}$, the maximal and minimal allowed length for mesh edges, respectively.

Note that $A^n$ and $Q^n$ are unknowns as they are the solution of the problem at time $t = t_n$; therefore, the metric tensor $M$ is approximated using the numerical solution at time $t = t_n$, $A^n, Q^n$. For more details and the extension to bidimensional domains, see [6,7]

4. CONSERVATIVE MESH ADAPTATION ALGORITHM

Once the metric tensor $M$ is computed, the mesh is adapted using an anisotropic Delaunay algorithm. For one-dimensional meshes, the algorithm is simple (see [6,7] $d_1$ be the length of the edge $a$, with respect to the metric tensor $M$.

Three possible cases can be distinguished:

- If $d_1 > d_{\text{max}}$ ($d_{\text{max}} \approx 1.4$), then $a$ is cut into two edges. The length of the new edges is computed, and they are split until the length of all the new edges is smaller than $d_{\text{max}}$.
- If $d_1 < d_{\text{min}}$ ($d_{\text{min}} \approx 0.6$), then the edge $a$ is suppressed. As this process implies that neighbour edges change, we must check if their lengths are larger than $d_{\text{max}}$. In that case, the previous step is applied to the corresponding edges.
- If $d_{\text{min}} \leq d_1 \leq d_{\text{max}}$, $a$ is kept.

One of the most difficult problems on mesh adaptation is the interpolation of the numerical solution onto the adapted mesh. This is critical if the studied phenomena are unsteady; a deficient interpolation could spoil the good properties of the numerical scheme, as conservation and monotonicity. The usual interpolation operator in mesh adaptation is the linear one, but this operator, in general, is not conservative. This means that, given the numerical solution of the shallow water equations over the mesh $T_h$ at step $n$, $(A^n, Q^n)$,

$$\int_{T_h} A^n \neq \int_{T_h} \Pi_{h'}[A^n], \quad \int_{T_h} Q^n \neq \int_{T_h} \Pi_{h'}[Q^n],$$

being $\Pi_{h'}[A^n], \Pi_{h'}[Q^n]$ a continuous piecewise linear interpolation of $(A^n, Q^n)$ over the adapted mesh, $T_{h'}$, at time $t = t_n$.

In order to guarantee the conservation of variables during the mesh adaptation process, we propose the following interpolation operator $\Pi_{h'}[A^n]$ is the continuous piecewise linear function over the mesh $T_{h'}$ such as

$$\Pi_{h'}[A^n](v'_j) = \frac{\int_{C'_j \cap T_h} A^n}{|C'_j|}, \quad (4)$$

where $v'_j$ is a vertex of $T_{h'}$, $C'_j$ is the cell associated to $v'_j$, and $|C'_j|$ is the length of cell $C'_j$.

Note that $\int_{C'_j \cap T_h} A^n$ could be difficult and expensive to calculate if $T_h \neq T_{h'}$. We can avoid this problem if we perform this computation during the mesh adaptation loop, since $C'_j \cap T_h$ is easily determined.

Using conservative interpolation together with mesh adaptation, the discretisation error and CPU time can be reduced substantially for unsteady problems (see Table 1).

5. NUMERICAL RESULTS

5.1. Dam Break Problem

This is an interesting problem to test the efficiency of conservative mesh adaptation algorithms for nonsteady flows with shocks since it has an analytical solution.

This problem is generated by the homogeneous one-dimensional shallow water equations with the initial conditions

$$h(x, 0) = \begin{cases} h_L, & \text{if } x \leq \frac{L}{2}, \\ h_R, & \text{if } x > \frac{L}{2}, \end{cases} \quad Q(x, 0) = 0.$$  

In this case, $h_L = 2$, $h_R = 1$, and $L = 60$ m.
Table 1 Dam break problem

<table>
<thead>
<tr>
<th>Adaptation</th>
<th>Scheme</th>
<th>N Nodes</th>
<th>$L^2$-Error</th>
<th>CPU Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>Roe</td>
<td>3000</td>
<td>0.053487</td>
<td>164.45 s</td>
</tr>
<tr>
<td>No</td>
<td>TVD-MC</td>
<td>1000</td>
<td>0.052360</td>
<td>23.3 s</td>
</tr>
<tr>
<td>Conservative</td>
<td>Roe</td>
<td>332</td>
<td>0.058669</td>
<td>16.16 s</td>
</tr>
<tr>
<td>Nonconservative</td>
<td>Roe</td>
<td>325</td>
<td>0.058308</td>
<td>16.20 s</td>
</tr>
</tbody>
</table>

Table 1 summarises the results obtained for the $Q$-scheme of Roe and the TVD-McCormack scheme (TVD-MC) with a uniform mesh and also for the $Q$-scheme of Roe with conservative and nonconservative adaptation at time $t = 2.5$ s. As can be observed, the $Q$-scheme of Roe + conservative adaptation only needs 332 nodes to obtain an error of 0.053 units. If a uniform mesh is used, the number of nodes must be about 3000 for a similar error. For a higher-order scheme as the TVD-MC, the number of nodes is about 1000. Note that, if a nonconservative mesh adaptation algorithm is used, the approximation error increases up to 0.0583 units. The reduction of CPU time for a similar tolerance error is significant, if mesh adaptation is used and note that the computational cost for conservative and nonconservative adaptation is practically the same.

Figure 1 shows a comparison between the numerical solution for the dam break problem with conservative mesh adaptation and the exact solution at time $t = 2.5$ s.

5.2. A Steady Flow in a Converging-Diverging Channel

In order to analyse the behaviour of conservative mesh adaptation, a classical problem has been selected: a transcritical case in a steady flow in a converging-diverging channel with flat bed. The width variation modifies the steady-state profiles, and due to the boundary conditions, a stationary hydraulic jump appears to connect subcritical and supercritical flows.

More precisely, the geometrical domain of the flow is an interval of $L = 500$ m with flat bed and a sinusoidal width variation given by

$$B(x) = \begin{cases} 
5 - 0.7065 \left(1 + \cos \left(2\pi \left(\frac{x - 250}{300}\right)\right)\right), & \text{if } |x - 250| \leq 150, \\
5, & \text{otherwise}
\end{cases}$$
Conservative Grid Adaptation Technique

Figure 2: Steady flow in a converging-diverging channel. Comparison between conservative adaptation and the TVD-MC scheme

Table 2: Steady flow in a converging-diverging channel

<table>
<thead>
<tr>
<th>Adaptation</th>
<th>Scheme</th>
<th>N Nodes</th>
<th>Error</th>
<th>CPU Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>No</td>
<td>Roe</td>
<td>3000</td>
<td>0.328287</td>
<td>8996 s</td>
</tr>
<tr>
<td>No</td>
<td>TVD-MC</td>
<td>550</td>
<td>0.194526</td>
<td>304 s</td>
</tr>
<tr>
<td>Conservative</td>
<td>Roe</td>
<td>443</td>
<td>0.194083</td>
<td>280 s</td>
</tr>
<tr>
<td>Nonconservative</td>
<td>Roe</td>
<td>444</td>
<td>0.195083</td>
<td>207 s</td>
</tr>
</tbody>
</table>

Subcritical initial conditions are stated at a depth $h(x, 0) = 2 \text{m}$. As boundary conditions, the discharge $Q(0, t) = 20 \text{cum/s}$ at the upstream and a 0.1 m high weir condition (see [2,3]) at the downstream boundary are imposed. As Figure 2 shows, the water accelerates as it approaches to the point of maximal contraction, the flow becomes critical there and it changes then to supercritical flow that gives rise to a stationary hydraulic jump to connect with the subcritical profile required by the downstream condition.

Figure 2 shows the numerical solution obtained with the TVD-MC scheme with a uniform mesh and with the Q-scheme of Roe with conservative adaptation. Table 2 summarises the numerical experiments for the different schemes with and without conservative adaptation. Note that, in this case, conservative and nonconservative adaptation give similar results and CPU time is considerably reduced when mesh adaptation is used.

5.3. Surge Propagation Through Converging-Diverging Channel

In this example, the geometrical domain for the flow is an interval of $L = 500 \text{m}$ with flat bed and a sinusoidal width variation given by

$$B(x) = \begin{cases} 5 - 0.7065 \left(1 + \cos \left(2\pi \frac{x - 250}{300}\right)\right), & \text{if } |x - 250| \leq 150, \\ 5, & \text{otherwise} \end{cases}$$

In this case, the exact solution cannot be obtained, and only comparisons with other schemes can be performed.

The time evolution of a surge is considered. A bore 9.79 m deep of 1000 cum/s propagates downstream over still water 1 m deep. A 2 m weir is supposed to be placed downstream. At time $t = 150 \text{s}$ the downstream end is reached by a front similar to the initial one so that it is partially reflected and partially transmitted over the weir (see Figure 3). Only 142 nodes are needed to
obtain a good approximation if conservative mesh adaptation is used, while a uniform mesh of 600 nodes is needed if TVD-McCormack is used. The reflected surge starts travelling upstream and it propagates until it becomes a stationary hydraulic jump in the contracting region. This final steady state is shown in Figure 4. The total CPU cost when using mesh adaptation is 186 seconds while the total CPU time when using TVD-McCormack is 360 seconds in a PENTIUM II (333 MHz).

6. CONCLUSIONS

Conservative mesh adaptation has been applied with success to 1D unsteady problems. The idea of the method is simple and it can be easily applied to other problems. The application to 2D and 3D configuration is straightforward.

Conservative interpolation guarantees the conservation of all the variables during mesh adaptation and, as the results show, the numerical error is reduced when it is used. The global CPU requirement is significantly reduced compared with a direct computation on a uniform fine mesh.
REFERENCES


