Piecewise-Linearized and Linearized $\theta$-Methods for Ordinary and Partial Differential Equations

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Abstract—Initial- and boundary-value problems appear frequently in many branches of physics. In this paper, several numerical methods, based on linearization techniques, for solving these problems are reviewed. First, piecewise-linearized methods and linearized $\theta$-methods are considered for the solution of initial-value problems in ordinary differential equations. Second, piecewise-linearized techniques for two-point boundary-value problems in ordinary differential equations are developed and used in conjunction with a shooting method. In order to overcome the lack of convergence associated with shooting, piecewise-linearized methods which provide piecewise analytical solutions and yield nonstandard finite difference schemes are presented. Third, methods of lines in either space or time for the solution of one-dimensional convection-reaction-diffusion problems that transform the original problem into an initial- or boundary-value one are reviewed. Methods of lines in time that result in boundary-value problems at each time step can be solved by means of the techniques described here, whereas methods of lines in space that yield initial-value problems and employ either piecewise-linearized techniques or linearized $\theta$-methods in time are also developed. Finally, for multidimensional problems, approximate factorization methods are first used to transform the multidimensional problem into a sequence of one-dimensional ones which are then solved by means of the linearized and piecewise-linearized methods presented here. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, several methods based on linearization techniques for solving initial- and boundary-value problems in ordinary and partial differential equations are reviewed. Some initial-value problems in ODEs can be solved using analytical methods that lead to exact solutions, but most of them can only be solved with numerical methods that yield an approximate solution. Among these numerical methods, it is possible to distinguish one-step methods like Euler's method, $\theta$-methods, or Runge-Kutta's methods, and multistep methods like Adams-Bashforth or Adams-Moulton techniques [1]. Related to the present paper are linearly implicit Runge-Kutta methods [2,3] that include Rosenbrock and W-methods. Rosenbrock's methods [4] are based on substituting the nonlinear terms in an implicit Runge-Kutta scheme by their linear approximation. W-methods...
are derived from Rosenbrock ones by approximating the Jacobian matrix. These methods have been studied by many researchers [3,5–7]. The linearized $\theta$-methods analyzed here are Rosenbrock and W-methods, but they have been obtained from $\theta$-techniques. Other authors such as Meyer-Spasche and coworkers [8–10] have been interested in the linearized trapezoidal rule which is the fully linearized $\theta$-method considered here with $\theta = 1/2$.

Other methods based on analytical approximations are Adomian's decomposition method and global linearization. Adomian's decomposition method [11,12] gives the exact solution of a nonlinear differential equation by means of a series that usually converges quickly. Every term of the series can be determined by using a recurrence formula that is obtained by expressing the nonlinear term in the differential equation as a sum of three operators: a nonlinear one, an easy-to-solve linear one, and the remaining linear part.

The use of a global linearization method is not justified if the equation is not weakly linear and the interval where the problem is solved is not sufficiently small. However, the piecewise-linearized methods considered here can be used in general situations because the linear approximation to the differential equation is only made in sufficiently small intervals.

In order to solve two-point boundary-value problems in ordinary differential equations, shooting methods, finite difference methods, or finite element methods can be used [13,14]. Shooting methods look for an initial value problem equivalent to the boundary-value problem to be solved, by first selecting an initial-value for the differential equation, then by solving the resulting problem, and finally, by testing if the solution satisfies the boundary conditions. If these conditions are satisfied, the problem is solved; if not, the initial values are modified and the iterative process is repeated. Finite difference methods transform the ordinary differential equation into an algebraic system whose solution approximates that of the differential problem in a grid of the domain. Finite element methods also transform the ordinary differential equation into an algebraic system, but whose solution is an approximate one in a finite functional space. In this paper, shooting methods and nonstandard finite difference techniques based on linearization are described.

Problems governed by partial differential equations have also been solved by means of finite element and finite difference methods. In a recent book, Mickens [15] analyzes nonstandard finite difference schemes to solve ordinary and partial differential equations. These schemes provide exact difference formulas for several differential equations and do not suffer from numerical instability. Mickens enumerates some conditions that these schemes must satisfy, but he does not describe a general method to design them. For some examples like the logistic equation, Mickens [16] deduces an exact finite difference scheme that can be expressed explicitly. However, as Agarwal indicated [17], Mickens gets the approximations to the difference equations using the known solution of the differential equation or by 'ad hoc' experimentation, so it is still necessary to develop a method in order to formulate nonstandard difference schemes.

Evolution problems can be solved by using methods of lines that discretize some but not all partial derivatives of the equation [18,19], and result in a system of ordinary differential equations or a system of partial differential ones in a smaller dimensional setting than the original one.

Operator-splitting techniques and fractional step methods [20] are usually employed to solve problems governed by partial differential equations that involve several spatial dimensions or different physical phenomena like reaction, diffusion, and/or convection. The expression 'fractional step method' means a set of methods that includes the ADI methods developed by Douglas, Peaceman, and Rachford, and LOD techniques developed by D'Yakonov, Yanenko, Marchuk, and Samarski. These methods provide an approximate solution to a difference or differential equation at a fixed instant, by performing several steps with different operators. In many cases, the intermediate schemes are obtained by means of an approximate factorization of the operator that defines the equation. For this reason, these methods are referred to here as approximate factorization techniques.

Many researchers have used fractional step techniques to solve differential equations. For example, Fairweather and Mitchell [21] solved the heat equation with Dirichlet conditions using
Peaceman and Rachford's method, showed that this method loses some accuracy if the boundary conditions depend on time, and proposed some modifications. Sommeijer et al. [22] tried to adapt these modifications to solve more general parabolic problems by means of several ADI and LOD methods, and used the method of lines in time with intermediate boundary conditions. These modifications are only useful if boundary conditions are of the Dirichlet type.

LeVeque and Oliger [23] considered a quasilinear, one-dimensional hyperbolic problem defined by a matrix that can be expressed as a sum of a matrix with small eigenvalues and another with large ones. Its solution is exponential and can be factorized with Strang's method [24]. This factorization is similar to an LOD method for multidimensional problems, but it corresponds to time splitting.

Sheng [25] proposed solving a parabolic differential equation in two dimensions by using a method of lines in time, i.e., by discretizing the spatial variables, so a differential system is obtained whose solution is exponential. This exponential solution can be factorized using Strang's method, but Sheng suggests using more terms in the factorization.

Operator-splitting methods are approximate factorization techniques which, instead of splitting the spatial variables (splitting in space), split some operators related to different physical phenomena (splitting in time). For example, when these methods are used to solve a nonlinear reaction-diffusion problem in one dimension [26,27], the reaction-diffusion operator is transformed into a sequence of reaction and diffusion operators, taking into account the characteristic reaction and diffusion times. The reaction operator defines a nonlinear, first-order differential equation and its discretization leads to a nonlinear algebraic system that is usually solved with the Newton-Raphson method until a convergence criterion is satisfied. The diffusion operator satisfies a partial differential equation that can be solved by means of a great variety of numerical and/or analytical techniques.

In most numerical studies concerning reaction-diffusion equations, the intermediate boundary conditions in operator-splitting techniques are assumed to coincide with the original boundary conditions. This hypothesis has sometimes been justified with a naive argument, reasoning that intermediate solutions obtained with operator-splitting techniques lack physical sense, and that the only valid solution is that obtained at the end of the computational cycle that involves the reaction and diffusion operators. However, intermediate boundary conditions are necessary for operator-splitting methods, and their lack of sense makes their selection more difficult. This topic has been studied by many investigators in the last few years [28–32] due to an increasing use of these methods in a great variety of problems arising in fluid dynamics, heat and mass transfer, climate and environment models, etc. In most cases, simple equations have been studied. For example, Khan and Liu [30] analyzed a linear, one-dimensional, convection-diffusion equation, using a Strang-type operator-splitting method that transforms the original convection-diffusion operator into a sequence of convection, diffusion, and convection operators which are solved with half, full, and half-time stepsizes, respectively, and proposed intermediate boundary conditions based on the Taylor's series expansion of the original boundary conditions.

Carpenter et al. [28] defined five sets of intermediate boundary conditions for the different stages of explicit Runge-Kutta methods in their studies of one-dimensional advection equations. These authors showed that intermediate boundary conditions based on physics include the original boundary conditions and their time derivatives, and coincide with the original ones if these do not depend on time.

Iskandar and Mohsen [29] combined linearization and operator-splitting techniques to build some algorithms for the one-dimensional Burgers' equation. Their operator-splitting techniques separate the nonlinear convection operator from the linear diffusion one, and these operators are solved in this order with a timestep of the same size as the full cycle one.

Steinthorsson and Shih [31] studied the approximate factorization errors due to several approximate factorization methods in order to reduce them in a three-dimensional, linear, advection equation.
More recently, Lanser and Verwer [32] analyzed a three-term symmetrical Strang operator-splitting method for a class of advection-diffusion-reaction problems from air pollution modelling and proposed several techniques to reduce the splitting error.

This paper has been organized as follows. In Sections 2 and 3, piecewise-linearized methods and linearized \( \theta \)-methods, respectively, for solving initial-value problems in ordinary differential equations are described, and some modifications to improve their computational efficiency such as Padé approximants, Schur's normal form, or partial linearization are suggested.

These methods can be used for solving two-point boundary-value problems with shooting techniques, but it is also possible to develop nonstandard finite difference methods by using piecewise linearization as shown in Section 4.

Methods for solving one-dimensional convection-reaction-diffusion problems in one dimension are discussed in Section 5. These problems can be solved by discretizing the time variable, and result in a two-point boundary-value problem at each time level. On the other hand, space discretization leads to initial-value problems in ODEs. It is also possible to use operator-splitting techniques and linearization methods for convection-reaction-diffusion problems.

In Section 6, it is shown that two-dimensional problems can be solved by discretizing the time variable, to obtain an elliptic partial differential equation at every time step which can be solved by means of approximate factorization techniques that lead to two-point boundary-value problems, or by discretizing the space coordinates resulting in initial-value problems in ODEs.

2. PIECEWISE-LINEARIZED METHOD FOR INITIAL-VALUE PROBLEMS IN ODES

Consider the initial-value problem

\[
x' = f(x, t),
\]

\[
x(t_0) = x_0,
\]

where \( x \in \mathbb{R}^n \), \( t \in \mathbb{R} \), \( f = (f_1, f_2, \ldots, f_n) \in C^2(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n) \), and the prime denotes differentiation with respect to \( t \).

Equation (1) is, in general, nonlinear, and therefore, may not be solved analytically, except in very special cases. However, if equation (1) is to be solved for \( t \in I = (t_0, T] \), an approximate, analytical solution may be obtained as follows. The interval \( I \) is first divided into \( n \) subintervals, i.e., \( I = (t_0, t_1] \cup \cdots \cup (t_{n-1}, t_n] \). In each interval, \( I_{i+1} = (t_i, t_{i+1}] \), the right-hand side of equation (1) is approximated by its Taylor's first-degree polynomial about \( (y(t_i), t_i) \) in \( I_i \).

\[
y' = g(y, t) \equiv F_i + J_i(y - y_i) + T_i(t - t_i), \quad t \in [t_i, t_{i+1}],
\]

\[
y(t_i) = y_i,
\]

where

\[
J_i = \frac{\partial f}{\partial x}(y(t_i), t_i), \quad T_i = \frac{\partial f}{\partial t}(y(t_i), t_i), \quad F_i = f(y(t_i), t_i),
\]

\( y(t_0) = x_0 \), and \( J_i \) is a Jacobian matrix. Note that, for \( f \in C^2 \), the linear approximation to \( f \) is accurate to \( O((t_{i+1} - t_i)^2) \). Furthermore, since equation (3) is linear, its analytical solution in \( I_i \), i.e., for \( t \in (t_i, t_{i+1}] \), may be written as

\[
y(t) = \int_{t_i}^{t} \exp(J_i(t - \tau))[T_i(\tau - t_i) + F_i] d\tau + y_i.
\]

If \( f(x, t) \) had been approximated by the Taylor's polynomial of zeroth degree, i.e., \( g(y, t) \equiv F_i \) in equation (3), its solution would be \( y(t) = F_i(t - t_i) + y(t_i) \) and the values of \( y(t_i) \) would correspond to those of the forward (explicit) Euler method.
Equation (6) provides an approximate, piecewise analytical solution to equation (1) where the value of \( y(t_i) \) depends on the solution to equation (3) in \( I_{i-1} \). Furthermore, since the right-hand side of equation (3) is a linear approximation to \( f(x,t) \), this method is referred to as a piecewise-linearized, analytical technique [33]. Note that if equation (1) were autonomous and \( f \) linear, \( f \) and its linear approximation would coincide, \( J_i \) would be a constant matrix, \( T_i = 0 \), and the solution to equation (1) would coincide with that to equation (3). Therefore, the piecewise-linearized method is exact for linear, autonomous, ordinary differential equations with constant coefficients and right-hand sides.

Equation (6) must be handled with care because it contains exponential functions whose evaluation may cause overflow, underflow, and/or loss of significant digits even for only a single ordinary differential equation. For systems of ordinary differential equations, equation (6) indicates that, in order to determine the approximate solution \( y(t) \) in each interval \( I_i \), some matrix exponentials must be calculated. Such exponentials may, in principle, be evaluated by means of the Cayley-Hamilton theorem which relates \( A^n \) to \( A^j \) where \( j = 0, 1, \ldots, (n - 1) \) through the characteristic polynomial of the matrix \( A \). However, such an evaluation would require an infinite number of operations. Other methods to determine the matrix exponential involve analytical techniques like Jordan’s form or Schur’s normal form, both of which are costly.

In order to reduce the computational cost associated with piecewise-linearized methods, an approximate triangular system, which is easier to solve, Padé approximations, or a partial linearization may be used. Note also that, in order to avoid many operations in the evaluation of the exponentials, the solution to equation (3) may be assumed to be a power series, i.e., \( y(t) = \sum_{j=0}^{\infty} a_j (t - t_i)^j \) where \( t \in (t_i, t_{i+1}] \), which, after substitution into equation (3) and matching equal powers of \( (t - t_i) \), yields the following recurrence formulae:

\[
\begin{align*}
  a_0 &= y_i, \\
  a_1 &= f_i, \\
  2a_2 &= J_i a_1 + T_i, \\
  ja_j &= J_i a_{j-1}, \quad j = 3, 4, \ldots.
\end{align*}
\]

The approximation to \( y(t) \) by means of a power series is essentially a Taylor’s series expansion of \( y(t) \) about \( t = t_i \) whose convergence is guaranteed provided that

\[
\limsup_{j \to \infty} \frac{\|a_j\|}{\|a_{j-1}\|} \Delta t_i < 1,
\]

and, taking into account equation (7), this is guaranteed if

\[
\|J_i\| \Delta t_i < 1,
\]

where \( \Delta t_i = t_{i+1} - t_i \).

The coefficients \( a_i \) may be determined by means of equation (7) until \( \|a_N(\Delta t_i)^N\| \leq \mu \), where \( \mu \) is a user-specified tolerance.

2.1. Piecewise-Linearized Methods with Variable Stepsize

In the previous section, it has been shown that the stepsize in piecewise-linearized methods must satisfy a restriction if the solution is expressed as a power series. This leads to a variable stepsize method which estimates the stepsize for the next iteration by using the following expression:

\[
\|J_i\| \Delta t_i = L_1 < 1,
\]

where \( L_1 \) is a specified constant. This method is referred to here as C1.
Another way to control the stepsize consists of bounding the error in the approximation to \( f(x,t) \). Using the Taylor’s series of \( f \) about \((x_i, t_i)\), this error is given by an expression which depends on the values of the partial derivatives at an undetermined point. Here, it is suggested to approximate these unknown derivatives by their values at \((x_i, t_i)\). In this way, the error can be approximated as

\[
\frac{\Delta t^2}{2} \max_{k=1,n} \left[ \sum_{r,s=1}^{N} \frac{\partial^2 f_k}{\partial x_r \partial x_s} (x_i, t_i) f_{i,r} f_{i,s} + \sum_{r=1}^{N} \frac{\partial^2 f_k}{\partial x_r \partial t} (x_i, t_i) f_{i,r} + \frac{\partial^2 f_k}{\partial t^2} (x_i, t_i) \right] = \epsilon, \tag{11}
\]

where \( f_{i,r} = f_r(x_i, t_i) \), which leads to another way to control the stepsize, referred to here as C2.

Shampine and Witt [19] proposed a strategy based on keeping constant the distance between two consecutive steps of the solution. This method can be used with piecewise-linearized methods by selecting the stepsize so that it is equal to the smallest positive root of the second degree equation

\[
\|F_i\| \Delta t_i + \|I_i + J_i F_i\| \Delta t_i^2 = L \tag{12}
\]

if it exists, and the resulting method is referred to here as C3. In this manner, the values that have already been calculated are taken into account.

These three strategies for determining the stepsize can fail if the factors that multiply \( \Delta t_i \) are zero or if there is not a positive solution of equation (12). In these cases, \( \Delta t_i \) is selected equal to \( \Delta t_{i-1} \).

2.2. Application: The Logistic Equation

The piecewise-linearized methods with constant and variable stepsize have been used to determine the solutions of many ordinary differential equations [33,34]. However, for conciseness, only the results obtained for the logistic equation are presented here.

Fisher’s or the logistic equation has been the subject of many studies [8,12] and is governed by the following differential equation:

\[
x'(t) = \lambda x(t)(1 - x(t)),
\]

\[
x(0) = x_0.
\]

The results obtained by means of the piecewise-linearized method with fixed \( \Delta t = 0.05 \) (referred to as PL) for the logistic equation, \( \lambda = 1 \) and \( x_0 = 2 \), are more accurate than those of other \( O(\Delta t^2) \) methods like the modified Euler method or the trapezoidal rule, although less accurate than the explicit fourth-accurate four-stage Runge-Kutta method [34].

Figure 1 shows the results obtained with the piecewise-linearized method for \( \lambda = 1 \), \( x_0 = 2 \), and fixed \( \Delta t = 0.05 \) (PL) and with the variable stepsize strategies C1–C3. It can be observed that C2 obtains the most accurate results, while C1 and C3 get larger absolute local errors than PL.

The number of subintervals and the mean absolute error of these adaptive stepsize methods are shown in Table 1. This table illustrates that, for this problem, the stepsize control based on keeping constant the difference between two successive time steps (C3) uses fewer subintervals than the other methods but results in the largest mean absolute error. The stepsize control based on approximation error C2 is the most accurate technique and uses fewer subintervals than the fixed stepsize method PL.

<table>
<thead>
<tr>
<th>Method</th>
<th>Subintervals</th>
<th>Mean Absolute Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>MC1</td>
<td>135</td>
<td>6.83E-5</td>
</tr>
<tr>
<td>MC2</td>
<td>188</td>
<td>1.89E-5</td>
</tr>
<tr>
<td>MC3</td>
<td>134</td>
<td>9.86E-5</td>
</tr>
<tr>
<td>PLM</td>
<td>200</td>
<td>4.37E-5</td>
</tr>
</tbody>
</table>
Piecewise-Linearized and Linearized $\theta$-Methods

3. LINEARIZED $\theta$-METHODS

A $\theta$-method to solve equation (1) can be expressed as

$$\frac{x_{n+1} - x_n}{\Delta t} = (1 - \theta)f(x_n, t_n) + \theta f(x_{n+1}, t_{n+1}),$$  \hspace{1cm} (13)

where $0 \leq \theta \leq 1$. This scheme is nonlinear if $f$ is nonlinear except for $\theta = 0$.

If, in equation (13), $f(x_{i+1}, t_{i+1})$ is linearized with respect to $t_i$, then the following linearized $\theta$-method results:

$$\Delta x_i = \Delta t_i f(x_i, t_i) + \Delta t_i (1 - \theta) \frac{\partial f}{\partial x}(x_i, t_i) \Delta x_i + \frac{\partial f}{\partial t}(x_i, t_i) \Delta t_i,$$  \hspace{1cm} (14)

which can be expressed as

$$\left[ I - \Delta t_i (1 - \theta) \frac{\partial f}{\partial x}(x_i, t_i) \right] \Delta x_i = \Delta t_i f(x_i, t_i) + \Delta t_i^2 (1 - \theta) \frac{\partial f}{\partial t}(x_i, t_i).$$  \hspace{1cm} (15)

In this way, a fully-linearized $\theta$-method for solving equation (1) leads to the following scheme:

$$\Delta x_i = \Delta t \Phi_f(t_i, x_i, \Delta t) = \Delta t_i A_i^{-1}(f_i + T_i \Delta t),$$  \hspace{1cm} (16)

where

$$A_i = I - \Delta t \theta J_i.$$  \hspace{1cm} (17)

These methods lead to a linear system of equations that must be solved at each step. This can be expensive because the coefficient matrix of the system is, in general, dense. So, other
linearized \( \theta \)-methods are proposed. Diagonally linearized \( \theta \)-methods replace the Jacobian matrix in equation (17) with a diagonal one with nonzero elements equal to the diagonal elements of the original Jacobian matrix. This provides a system of uncoupled equations at each time step

\[
x_{i+1} = x_i + \Delta t E_i (f_i + \Delta t T_i),
\]

where

\[
E_i = [I - \Delta t \theta \text{diag}(J_i)]^{-1},
\]

and \( \text{diag}(J_i) \) is a diagonal matrix whose nonzero elements are \( \frac{\partial f_k}{\partial x_i} (x_i, t_i) \).

Upper and lower triangularly linearized \( \theta \)-methods can be obtained by approximating the Jacobian matrix by its upper and lower triangular approximations, respectively, i.e.,

\[
x_{i+1} = x_i + \Delta t E_i (f_i + \Delta t T_i),
\]

where

\[
E_i = [I - \Delta t \theta A_i]^{-1},
\]

and \( A_i = \text{upp}(J_i) \) or \( A_i = \text{low}(J_i) \), respectively.

Fully linearized \( \theta \)-methods are A-stable for the same values of \( \theta \) as classical implicit \( \theta \)-methods. Furthermore, both methods have the same linear stability function

\[
r(z) = \frac{1 + \theta z}{1 - (1 - \theta)z}.
\]

If \( f \in C^2 \) and its second partial derivatives are bounded, linearized \( \theta \)-methods are convergent. Their order is 2 if \( \theta = 0.5 \) and full linearization is considered, and 1, otherwise.

Note that \( N! \) different triangularly linearized \( \theta \)-methods can be obtained depending on the order in which the triangular linearization is carried out. Note also that partially linearized \( \theta \)-methods are sequentially implicit methods which provide analytical solutions.

3.1. Linearized \( \theta \)-Methods vs. Rosenbrock’s Methods and W-Methods

Classical linearly implicit methods for initial-value problems in ODEs [2] can be expressed as

\[
y_{n+1} = y_n + \Delta t \sum_{i=1}^{s} b_i y'_{n_i},
\]

\[
y_{ni} = y_n + \Delta t \sum_{j=1}^{i-1} \alpha_{ij} y'_{nj},
\]

\[
y'_{ni} = f(y_{ni}, t_n + \alpha_i \Delta t) + \Delta t J_n \sum_{j=1}^{i} \gamma_{ij} y_{nj} + \Delta t \gamma_i g_n.
\]

Rosenbrock’s methods are linearized Runge-Kutta schemes and can be expressed as equation (23) where \( J_n \) corresponds to the Jacobian matrix of \( f \), and \( g_n \) is the vector with the first derivatives of \( f \) with respect to \( t \). W-methods are linearly implicit methods with the only restriction that \( J_n \) is an approximation to the Jacobian matrix of \( f \). Fully and partially linearized \( \theta \)-methods have been obtained by linearizing \( \theta \)-methods, but it has been shown [34] that they are Rosenbrock’s and W-methods, respectively.
3.2. Application: The Logistic Equation

The linearized $\theta$-methods have been used to solve the logistic equation and the results can be seen in Figure 2 for $\theta = 0$, 0.5, and 1 referred to as THL0 (explicit Euler method), THL2, and THL1, respectively. This figure shows that THL2 yields more accurate results than the iterative, classical $\theta$ method with $\theta = 0.5$ (THI2), and that THL1 and THL0 are less accurate than THL2 and THI2. Note that this is in accordance with the $O(\Delta t)$ accuracy of THL1 and THL0.

Figure 2 also shows that THL2 yields local errors of the same magnitude and trends as those of PL.

Linearized $\theta$-methods applied to the logistic equation results in

$$ x_{n+1} = x_n + \frac{\Delta t \lambda x_n (1 - x_n)}{1 - \Delta t \lambda (1 - 2x_n)}. $$

This scheme can only be applied if the denominator in the right-hand of the equation is not zero, i.e., if $1 - \Delta t \lambda (1 - 2x_n) \neq 0$, except if the numerator is also zero, because in this case, there may be a removable singularity. It is possible to prevent this situation by changing the stepsize $\Delta t$. Note that the equilibrium points of this scheme are the same as those of the original equation.

More examples can be seen in [34,35] where systems of differential equations have been solved with fully and partially linearized methods. Other examples of linearized $\theta$-methods will be given below when analyzing partial differential equations.

4. PIECEWISE-LINEARIZED METHODS FOR TWO-POINT BOUNDARY-VALUE PROBLEMS IN ODES

Consider the following two-point boundary-value problem:

$$ y'' = f(x, y, y'), \quad x \in (a, b), $$

$$ y(a) = y_0, \quad y(b) = y_f, $$

Figure 2. Logistic equation. Solution and local errors of linearized $\theta$-methods.
where \( x \in [a, b] \), \( x_0 = a < x_1 < \cdots < x_{n-1} < x_n = b \), and the primes denote differentiation with respect to \( x \).

An easy way to solve this problem by means of linearized methods is to combine them with shooting techniques that reduce the two-point boundary-value problem to an initial-value one. First, the interval \([a, b]\) is divided into \( n \) subintervals. In each subinterval \( I_{i+1} = [x_i, x_{i+1}] \), equation (24) is approximated by

\[
y''(x) = f_i + X_i(x - x_i) + G_i(y - y_i) + H_i(y' - y'_i), \quad x \in (x_i, x_{i+1}),
\]

(26)

where \( G_i = \frac{\partial f}{\partial y} (x_i, y_i, y'_i) \), \( H_i = \frac{\partial f}{\partial y'} (x_i, y_i, y'_i) \), \( X_i = \frac{\partial f}{\partial x} (x_i, y_i, y'_i) \), \( f_i = f(x_i, y_i, y'_i) \), and the following initial conditions are considered:

\[
y_{i+1}(x_i) = y_i = y_{i_i}(x_i),
\]

\[
y'_{i+1}(x_i) = y'_i = y'_{i_i}(x_i).
\]

(27)

Then, one looks for a root of

\[
r(s) = y_s(b) - y(b) = 0,
\]

(28)

where \( y_s \) is the solution of equation (26) obtained with a piecewise-linearized method using initial conditions \( y_s(a) = y_a, y'_s(a) = s \). A Newton-Raphson method for \( r(s) = 0 \) leads to

\[
s^k = s^{k-1} - \frac{r(s^{k-1})}{r^*(s^{k-1})},
\]

(29)

where

\[
r^*(s) = \frac{r(s + \Delta s) - r(s)}{\Delta s}.
\]

(30)

If the problem is determined by a linear equation, i.e., if \( f \) is given by

\[
f(x, y, y') = A(x) + B(x)y + C(x)y',
\]

(31)

it is possible to use the principle of superposition to avoid solving the nonlinear equation (28). In this case, the method consists of solving the differential equation

\[
y''_i = A(x) + B(x)y_i + C(x)y'_i,
\]

(32)

with initial conditions \( y_i(a) = y_0, y'_i(a) = y'_{0i} \), where \( i = 1, 2 \), \( y'_{01} \neq y'_{02} \), and the linear algebraic system

\[
c_1 + c_2 = 1,
\]

\[
c_1 y_1(b) + c_2 y_2(b) = y_f,
\]

which has just one solution if \( y_1(b) \neq y_2(b) \).

This method gives the exact solution if \( A(x) \) is a linear function of \( x \), and \( B \) and \( C \) are constant; otherwise, the error in the approximation is \( O(\Delta x^2) \) [36].

It is also possible to solve analytically the following linear ordinary differential equation obtained from equation (24) by replacing the nonlinear term \( f \) by its Taylor’s first-degree polynomial around \( x_i \):

\[
y''(x) = f_i + X_i(x - x_i) + G_i(y - y_i) + H_i(y' - y'_i), \quad x \in (x_i, x_{i+1}),
\]

(33)

with boundary conditions \( y(x_i) = y_i \) and \( y(x_{i+1}) = y_{i+1} \). The continuity of the solution and/or its derivative at the inner points \( x_i, i = 1 \ldots n - 1 \), leads to a nonlinear system with unknowns \( y_i, i = 1 \ldots n - 1 \).
The solution of the linear differential equation (33) is

\[ y = y_H + y_P, \]  

where \( y_H \) depends on \( d_i \), the discriminant of the second degree equation

\[ r^2 - H_i r - G_i = 0. \]  

The particular solution of the nonhomogeneous equation depends on \( f_i, X_i, H_i, \) and \( G_i \).

A nonstandard finite difference method is obtained by matching these solutions in two adjacent subintervals at \( x_i \). For example, if \( d_{i-1} > 0, \) and \( G_i, G_{i-1} \neq 0, \) this method leads to

\[ \begin{align*}
-x_i + C^1 r_i^+ + C^2 r_i^- &= -x_{i-1} + C^1 r_{i-1}^+ \exp (r_{i-1}^+ h_{i-1}) + C^2 r_{i-1}^- \exp (r_{i-1}^- h_{i-1}).
\end{align*} \]  

This equation is nonlinear because \( X_i, G_i, \) and \( r_i^\pm \) depend on \( y_i \) at each \( i \). Therefore, \( n - 1 \) equations and \( 2n - 1 \) unknowns \( (y_i, y_i') \) are obtained. \( y_i' \) can be determined using the analytical expression of the solution in each subinterval \( I_i, \)

\[ y_i' = -\frac{x_i}{G_i} + C^1 r_i^+ + C^2 r_i^- . \]  

The above expressions have been obtained by linearizing equation (24) by means of the Taylor’s series of \( f \) around the left point of the interval \( I_i, \) i.e., by means of a ‘left’ expansion. In an analogous way, a ‘right’ expansion can be considered by linearizing equation (24) about the right point of \( I_i. \) Another possibility is to consider both left and right expansions and use their arithmetic mean as an approximate solution, or perform the expansions with respect to an interior point of \( (x_{i-1}, x_i) \).

The above methods lead to continuously differentiable approximations to the original problem and are referred to here as \( C^1 \) techniques.

Instead of considering the subinterval \( (x_{i-1}, x_i) \), one may consider the subinterval \( (x_{i-1}, x_{i+1}) \), employ a Taylor’s zeroth-degree polynomial in the expansions around \( x_i \), and require continuity of the solution at all interior \( x_i \); the resulting technique is referred to here as a \( C^0 \) method.

By employing the techniques described in previous paragraphs, nonstandard, nonlinear, three-point, finite difference formulas can be obtained; these techniques provide differentiable solutions without interpolation and the exact solution for linear differential equations with both constant coefficients and nonhomogeneous terms that are linear functions of \( x \). Note that some coefficients in the above schemes depend on \( y_i \) and \( y_i' \), so an iterative method that uses initial values \( y_0, y_0' \) and calculates new values \( y_1, y_1' \) must be employed. This process is repeated until the difference between the results at two successive iterations is sufficiently small. Iterations are not needed for linear equations because the principle of superposition can be applied.

4.1. Application I: A Linear Perturbation Problem

As an illustration, the following linear perturbation problem has been studied:

\[ \begin{align*}
&ey'' - y = 0, \\
y(-1) = 1, &\quad y(1) = 2,
\end{align*} \]

whose analytical solution is

\[ y(t) = \frac{\left(2e^{1/\sqrt{\xi}} - e^{-1/\sqrt{\xi}}\right) e^{x/\sqrt{\xi}} + \left(e^{1/\sqrt{\xi}} - 2e^{-1/\sqrt{\xi}}\right) e^{-x/\sqrt{\xi}}}{e^{2/\sqrt{\xi}} - e^{-2/\sqrt{\xi}}}. \]
This solution has two boundary layers whose thickness decreases as the value of \( \epsilon \) is decreased. The problem has been solved for \( \epsilon = 10^{-1} \) and \( 10^{-2} \) by using shooting techniques together with the fourth-order explicit Runge-Kutta method (SHR), the modified Euler method (SHE), and the piecewise-linearized method (SHL) described above. The principle of superposition with the piecewise-linearized method can also be used because the problem is linear and is referred to as PL. The methods PL and SHL use just two subintervals and obtain absolute local errors smaller than SHE and SHR. This is not surprising because the problem is linear with constant coefficients and PL should obtain the exact solution in the absence of round-off errors.

For \( \epsilon = 10^{-3} \) and \( 10^{-4} \), the shooting methods considered in this paper do not converge, because the determination of the slope \( s \) at the left boundary is an ill-conditioned problem; we, therefore, employ a finite difference and a finite element method to solve the problem. The finite difference method employed here is referred to as FD and uses central differences to discretize the second-order derivative with an accuracy of \( O(\Delta x^2) \). This method yields a tridiagonal linear system of algebraic equations which may be solved by means of Thomas' algorithm [37]. Here, a fixed stepsize \( \Delta x = 10^{-4} \) is employed.

The finite element method is denoted by FE and is based on \( C^0 \)-linear Lagrange elements, i.e., the boundary-value problem is approximated in the finite-dimensional space of the functions which are linear in every subinterval \([x_{i-1}, x_i]\), where \( x_i = -1 + i\Delta x \), using a constant stepsize \( \Delta x = 10^{-4} \). This method yields a tridiagonal linear system which is solved by using the same technique as the finite difference method.

Figure 3 shows the results obtained with FD, FE, and PL for the problem considered and \( \epsilon = 10^{-4} \). Note that the local errors of PL are lower than those of the finite element and finite difference techniques. Note also that FD and FE yield the same local errors.
4.2. Application II: A Nonlinear Problem

Nonlinear two-point boundary-value problems can be solved by means of shooting techniques and nonstandard finite difference methods C0 and C1 described previously. As an illustration, the following problem has been analyzed:

$$y'' = 1 + (y')^2, \quad y(0) = 0, \quad y\left(\frac{\pi}{4}\right) = \frac{\ln 2}{2},$$

whose exact solution is

$$y(x) = -\ln(\cos(x)).$$

Figure 4 shows the results with SHE, SHR, SHL, C0, and C1 when 100 subintervals are used in the discretization. SHL obtains the smallest local errors of $O(10^{-6})$, C0, and C1 lead to local errors of $O(10^{-4})$, and SHE and SHR have almost the same errors which are larger than those of the other methods presented in the figure. However, if 50 subintervals are considered, the shooting methods do not converge, but C0 and C1 yield local errors of $O(10^{-3})$ and $O(10^{-4})$, respectively.

Other two-point boundary-value problems that have been solved by means of methods based on linearization can be found in [34,36].
5. ONE-DIMENSIONAL
CONVECTION-REACTION-DIFFUSION PROBLEMS

A one-dimensional convection-reaction-diffusion problem can be written as

\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} + S(u, x, t), \]
\[ u(x, 0) = u_0(x), \quad x \in [0, L], \]
\[ u(0, t) = \alpha(t), \quad u(L, t) = \beta(t), \quad t \in [0, T], \]

where \( u, v, \) and \( D \) are functions defined in a domain of \( \mathbb{R}^2 \) over \( \mathbb{R} \). In order to solve this problem numerically with linearized methods, several techniques can be used. First, methods of lines in space lead to two-point boundary-value problems which can be solved by using shooting and linearization or the finite differences methods \( C_0 \) and \( C_1 \) described in Section 4. Second, methods of lines in time result in initial-value problems which can be solved by using the piecewise-linearized or linearized \( \theta \)-methods described in Sections 2 and 3, respectively. Finally, operator-splitting techniques transform a convection-reaction-diffusion problem into a sequence of reaction, diffusion, and convection ones that can be solved with methods based on linearization.

5.1. Methods of Lines in Space

Consider a one-dimensional convection-reaction-diffusion problem with Dirichlet conditions, and \( v \) and \( D \) constant (\( D \neq 0 \)). Methods of lines in space transform the partial differential equation in a system of ordinary differential equations by discretizing the time variable. In this way, a partition of \([0, T]\) into subintervals \([t_n, t_{n+1}]\), \( n = 0, \ldots, N \) is considered. Then, the derivative of \( u \) with respect to \( t \) is approximated by

\[ \frac{\partial u}{\partial t}(x, t_{n+1}) = \frac{u(x, t_{n+1}) - u(x, t_n)}{\Delta t_n} + O(\Delta t_n), \]

where \( \Delta t_n = t_{n+1} - t_n \), and the following two-point boundary-value nonlinear problem results at each \( t_{n+1} \):

\[ \frac{d^2 u}{dx^2}(x) = f\left(t_{n+1}, u_n, x, u, \frac{du}{dx}\right), \]
\[ u(0) = \alpha(t_{n+1}), \quad u(L) = \beta(t_{n+1}), \]

where

\[ f\left(t_{n+1}, u_n, x, u, \frac{du}{dx}\right) = \frac{v}{D} \frac{du}{dx} + \frac{1}{D \Delta t} (u - u_n)r - \frac{1}{D} S(u, x, t_{n+1}). \]

This boundary-value problem can be solved with the methods described in Section 4. Shooting methods may not converge, so it is preferable to use the nonstandard finite difference methods \( C_0 \) and \( C_1 \) [38].

5.2. Methods of Lines in Time

Another manner to solve equation (38) is to use a method of lines in time together with linearized methods for initial-value problems as follows. First, one discretizes the spatial derivatives while time is kept continuous. For example, the following \( O(\Delta x^2) \) finite difference operators can be used:

\[ \Delta^0 u_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}, \]
\[ \delta^2 u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}. \]
Spurious oscillations and numerical instabilities [39] can result when equation (45) is employed and the velocity \( v \) is large or the grid spacing is not sufficiently small, but can be eliminated by using nonsymmetric \( O(\Delta x) \) operators for the convection term or by introducing numerical diffusion terms in the discretized equation [40,41].

Using operators \( \Delta^0 \) and \( \delta^2 \), the discretization of the spatial derivatives in equation (38) leads to

\[
\frac{du_i}{dt} - v_i \Delta^0 u_i = D_i \delta^2 u_i + S(u_i, x_i, t), \quad i = 1, \ldots, M - 1, \tag{47}
\]

\[
u_i(0) = u_0(x_i), \tag{48}
\]

that can be expressed as a system of first-order, linear, ordinary differential equations as

\[
\frac{dU}{dt} = F(U, t), \tag{49}
\]

\[
U(0) = U_0, \tag{50}
\]

where \( U = (u_1, u_2, \ldots, u_{M-1})^T \), \( F = (F_1, \ldots, F_{M-1})^T \), \( F_j(U, t) = v_j \Delta^0 u_j + D_j \delta u_j + S(u_j, x_j, t) \), for \( i = 1, \ldots, M - 1 \) and \( U_0 = (u_0(x_1), u_0(x_2), \ldots, u_0(x_{M-1})) \).

Equation (49) can be solved with the methods described in Sections 2 and 3 as follows.

### 5.2.1. Continuous-time methods

Piecewise-linearized methods can be used to solve the above problem resulting in the following linear differential system:

\[
\frac{dU}{dt} = F^n + TF^n(t - t_n) + JF^n(U - U^n), \quad t \in [t_n, t_{n+1}], \tag{51}
\]

where \( JF^n \) is a tridiagonal matrix and the error incurred when equation (49) is approximated by equation (51) is \( O(\Delta t) \).

It is possible to use the Schur’s normal form of \( JF_n \) to get an equivalent triangular system, but this is computationally very expensive. It is computationally more efficient to perform a partial linearization instead of a full linearization or to consider Padé approximants to approximate the matrix exponential that appears in the solution of equation (51).

It is also possible to perform the linearization of the nonlinear term before the spatial discretization, and obtain a linear approximation to the nonlinear partial differential equation which results in a piecewise-linear partial differential equation which can be solved analytically although at a high cost.

### 5.2.2. Time discretization

Equation (47) can be solved by discretizing the time derivative using classical iterative or linearized \( \theta \)-methods. Iterative, implicit, \( \theta \)-methods lead to

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \theta v_i^{n+1} \Delta^0 u_i^{n+1} + (1 - \theta) v_i^n \Delta^0 u_i^n + \theta D_i^{n+1} \delta^2 u_i^{n+1} + (1 - \theta) D_i^n \delta^2 u_i^n + \theta S_i^{n+1} + (1 - \theta) S_i^n, \tag{52}
\]

i.e., a nonlinear algebraic system, whose solution is computationally expensive. Instead, fully-linearized, implicit, \( \theta \)-methods result in

\[
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \left[ \theta v_i^{n+1} \Delta^0 u_i^{n+1} + (1 - \theta) v_i^n \Delta^0 u_i^n \right]
+ \left[ \theta D_i^{n+1} \delta^2 u_i^{n+1} + (1 - \theta) D_i^n \delta^2 u_i^n \right] + S_i^n + \theta T_i^n \Delta t + \theta J_i^n \left( u_i^{n+1} - u_i^n \right), \tag{53}
\]
i.e., a tridiagonal linear algebraic system, with a truncation error equal to $O(\Delta t, \Delta x^2)$ if $0 \leq \theta \leq 1$, \(\theta \neq 0.5\), and $O(\Delta t^2, \Delta x^2)$ if $\theta = 0.5$. This equation coincides with the Briley and McDonald's one [41] if $S$ does not depend on $t$. Note that equation (53) is a tridiagonal system, so it can be solved by means of Thomas' method [37].

Equation (53) can be expressed in a delta formulation as

$$\frac{\Delta u_i}{\Delta t} = v_i \left[ \theta \Delta^0 u_i + \Delta^0 u^n_i \right] + D_i \left[ \theta \Delta^1 u_i + \Delta^1 u^n_i \right] + J^n_i + \theta T^n_i \Delta t + \theta J^n_i \Delta u_i,$$

where $\Delta u_i = u^{n+1}_i - u^n_i$. This equation coincides with that of Beam and Warming [40] if $S$ does not depend on $t$.

Partially linearized $\theta$-methods are not recommended for discretizing equation (47) which corresponds to a single reaction-diffusion equation, and therefore, the linearization ought to be performed with respect to the nodal values of $u$, i.e., $u_i$, because the increase in efficiency does not, in general, compensate for the lower accuracy of these methods.

### 5.3. Operator-Splitting Techniques: Intermediate Boundary Conditions

Problems with different physical phenomena characterized by different time and/or space scales like reaction-diffusion problems can be solved using operator-splitting techniques. As an example, consider the following problem:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u, x, t). \quad (55)$$

Operator-splitting methods lead to a scheme like, e.g.,

$$u^{n+1} = (L_D)^M ((L_R)^N (u^n)),$$

where $M$ and $N$ are, respectively, the times that $L_D$ and $L_R$ are applied and

$$L_D : \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad L_R : \frac{\partial u}{\partial t} = f(u, t, x). \quad (57)$$

Other sequences of reaction and diffusion operators different from that of equation (56) may also be employed.

The stepsizes used with $L_D$ and $L_R$ can be different, especially if the equation models chemical reactions where the reaction is faster than diffusion.

The solution at each time level is obtained after a complete computational cycle. The intermediate solutions have no physical sense, although the main difficulty is to determine the appropriate intermediate boundary conditions required by the intermediate operators. The intermediate boundary conditions in operator-splitting techniques for linear problems can be determined analytically as follows.

Consider the following equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Au, \quad (x, t) \in [0, 1] \times [0, T], \quad (58)$$

$$u(x, 0) = u_0(x), \quad x \in [0, 1], \quad (59)$$

$$u(0, t) = f(t), \quad u(1, t) = g(t), \quad (60)$$

where $A$ is a constant. This equation can be solved analytically by means of the method of separation of variables.

The determination of the intermediate boundary conditions for $M = N = 1$ can be done by requiring that the solution obtained by means of operator-splitting coincides with the analytical solution at each time level.
Taking into account that the diffusion problem is
\[
\frac{\partial u^D}{\partial t} = \frac{\partial^2 u^D}{\partial x^2}, \quad (x, t) \in [0, 1] \times [0, \Delta t],
\]
(61)
\[ u^D(0, t) = F(t), \] (62)
\[ u^D(1, t) = G(t), \] (63)
\[ u^D(x, 0) = u_0(x); \] (64)
and the reaction problem is
\[
\frac{du^R}{dt} = Au^R, \quad (x, t) \in [0, 1] \times [0, \Delta t],
\]
(65)
\[ u^R(x, 0) = u^D(x, \Delta t), \] (66)
the solutions obtained with these two techniques (separation of variables and operator-splitting method) are equal at each time level if
\[
F(t) = \exp(-At)f(t), \]
(67)
\[ G(t) = \exp(-At)g(t). \] (68)

Linearized operator-splitting techniques can be used to solve nonlinear problems like
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + S(u), \quad u \in R,
\]
(69)
\[ u(x, 0) = u_0(x), \] (70)
\[ u(0, t) = f(t), \] (71)
\[ u(1, t) = g(t). \] (72)

In this case, the analytical solution of diffusion problem, \( u^D \), can be calculated as in the linear case, and a piecewise-linearized method can be used to solve
\[
\frac{du^R}{dt} = S(u^D) + J^D(u^R - u^D). \] (73)

5.4. Systems of One-Dimensional Partial Differential Equations

Consider the following system of convection-reaction-diffusion equations:
\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = D \frac{\partial^2 u}{\partial x^2} + S(u, x, t),
\]
(74)
\[ u(x, 0) = u_0(x), \quad x \in [0, L], \] (75)
\[ u(0, t) = \alpha(t), \quad u(L, t) = \beta(t), \quad t \in [0, T], \] (76)
for which methods of lines in space lead to a system of second-order ordinary differential equations which can be solved with either shooting and linearization methods or with a nonstandard finite difference scheme. A nonstandard finite difference scheme can be deduced in a similar way to those corresponding to a single convection-reaction-diffusion equation, but involve systems of larger size; so, it is recommended to use partial linearization instead of full linearization, because the loss of accuracy may be compensated by faster computation.

Methods of lines in time yield a system of first-order ordinary differential equations which can be solved by means of fully and partially piecewise-linearized techniques or fully and partially linearized \( \theta \)-methods. Of course, larger systems than for a single partial differential equation are now obtained, but these systems are usually determined by a nondense coefficient matrix, so they may be solved by means of iterative methods. Diagonal linearization of \( S \) allows us to solve an uncoupled system for every component of \( u \), but incurring a loss of accuracy. A triangular linearization of \( S \) leads to a sequentially coupled system for the components of \( u \) with a smaller loss of accuracy.
5.5. Application: A One-Dimensional Reaction-Diffusion Problem

The following system of one-dimensional reaction-diffusion equations proposed by Twizell et al. [42] has been solved by means of the linearized methods described in this section:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - uv, \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (u - K)v,
\end{align*}
\]

\[u(x, 0) = 0, \quad v(x, 0) = \exp(-x^2),\]

\[\frac{\partial u}{\partial x}(0, t) = \frac{\partial v}{\partial x}(0, t) = 0,\]

\[u(200, t) = 1, \quad v(200, t) = 0,\]

with \(K = 0.5\).

Figure 5 shows the solution obtained with a method that combines a three-point, fourth-order accurate spatial discretization of the compact or Hermitian type [43] with an explicit fourth-
order Runge-Kutta method for solving the ordinary differential system that results after spatial discretization. The two variables $u$ and $v$ become propagating fronts. The initial spike value of $v$ is rapidly diffused, while $u$ generates a front that it propagates towards greater values of $x$.

The method of lines in space together with the nonstandard finite difference method $C_1$ yields the local error shown in Figure 6. The largest errors are located in the fronts of $u$ and $v$ and are $O(10^{-3})$. Although not shown here [34], similar trends, but larger errors, have been obtained if the nonstandard finite difference method $C_0$ is employed, although this method is of exponential type and only requires continuity of the solution at the interior grid points, whereas $C_1$ requires smoothness at all the interior points. If the nonlinear term in Twizell's equation is partially linearized, the errors are approximately twice the local errors of the fully linearized technique if the linearization is triangular, and nearly ten times greater if the linearization is diagonal.

The method of lines in space together with linearized $\theta$-methods have also been used to solve Twizell's problem, and the results are shown in Figure 7 for $t = 100$ and $\theta = \beta = 0.5$ where $\theta$ and $\beta$ are the implicitness parameters for the reaction and diffusion operators, respectively.

The methods, whose local errors are shown in Figure 7, are identified by means of letters and numbers. The letters indicate the method used in the time discretization: $I$ and $L$ correspond to classical iterative or fully-linearized $\theta$-methods, respectively; $LD$ indicates diagonal linearization; $LUV$ and $LVU$ denote triangular linearization when equations are solved in the sequence $u \rightarrow v$. 

---

Figure 6. Local errors of $C_1$ for Twizell's one-dimensional reaction-diffusion problem.
and \( v \rightarrow u \), respectively; and \( \text{OS} \) refers to as an operator-splitting technique that solves the reaction operator with classical iterative \( \theta \)-methods. The number 2 or 4 indicates that the spatial discretization is performed by using \( O(Ax^2) \) or \( O(Ax^4) \) finite difference operators.

As can be observed in Figure 7, the greatest errors are located in the fronts of \( u \) and \( v \), are nearly constant before the front of \( u \), and decrease rapidly from the fronts of both \( u \) and \( v \); in fact, these errors are nearly zero for \( x \) greater than 160, so the plot in logarithm scale is discontinuous. The errors of \( L2 \) and \( L4 \) nearly coincide, and are greater and smaller, respectively, than those of \( \text{OS}4 \) and \( L4 \), respectively; furthermore, \( L4 \) is slightly more accurate than \( L4 \). The smaller accuracy of \( \text{OS}4 \) with respect to \( L2 \) is due to a certain decoupling between the reaction and diffusion processes.

Although not shown here [34,44], the errors are greater if other values of \( \theta \) or \( \beta \) are used, i.e., 0 or 1, as can be expected since the discretization error is \( O(\Delta t^2) \) if \( \theta = \beta = 1/2 \), but \( O(\Delta t) \) otherwise. Full linearization yields more accurate results than partial linearization, even if operator-splitting techniques are employed. Moreover, diagonally linearized \( \theta \)-methods are less accurate than triangularly linearized ones, while the accuracy of the latter is nearly independent of the order in which the dependent variables are solved, except at the fronts of both \( u \) and \( v \).
It has been observed that the accuracy of operator-splitting methods with linearization decreases as \( \Delta x \) increases; however, for \( \Delta x = 0.05 \), the errors for \( \Delta t = 0.5 \) and 0.25 are nearly the same.

Figure 8 illustrates the different results obtained by combining operator-splitting techniques with piecewise-linearized techniques and linearized \( \theta \)-methods. The methods are represented by a sequence of letters and numbers; those that begin with T correspond to a piecewise-linearized method, and the others to linearized \( \theta \)-methods. The letter D denotes diagonal linearization, and UV and VU indicate triangular linearization in the sequence \( u \rightarrow v \) and \( v \rightarrow u \), respectively. The number 4 is related to the \( O(\Delta x^4) \) accuracy of the spatial discretization.

As can be seen in Figure 8, linearized \( \theta \)-methods with operator-splitting are just a little less accurate than the piecewise-linearized method at the front of \( u \). The opposite behaviour is observed between the fronts and the right boundary where the errors of the linearized \( \theta \)-method are nearly zero for \( x \geq 150 \). The errors in \( v \) are nearly equal for the two linearization techniques.

6. TWO-DIMENSIONAL
CONVECTION-REACTION-DIFFUSION PROBLEMS

To solve a convection-reaction-diffusion problem in two dimensions, similar methods to those described above can be used together with a factorization technique.
As an example, consider

\[
\frac{\partial u}{\partial t} + \sum_{i=1}^{d} v_i \frac{\partial u}{\partial x_i} = \sum_{i=1}^{d} D_i \frac{\partial^2 u}{\partial x_i^2} + f(u, x, t), \quad x \in \Omega, \quad t \in (0, T],
\]

\[
u(x, 0) = u_0(x), \quad x \in \Omega,
\]

\[
u(x, t) = \alpha(x, t), \quad x \in \partial \Omega, \quad t \in [0, T],
\]

where \( \Omega \) is a rectangular domain in \( \mathbb{R}^d \) and \( v_i \) and \( D_i \) are smooth functions with domain in \( \mathbb{R}^d \times \mathbb{R} \) and image in \( \mathbb{R} \).

In this section, we describe several methods for the numerical solution of equations (77)–(79). A method of lines in space with an approximate factorization leads to a system of second-order ordinary differential equations which can be solved by means of shooting, linearization, or the finite differences methods C0 and C1.

A method of lines in time results in a system of first-order ordinary differential equations which can be solved by means of either the piecewise-linearized or the linearized \( \theta \)-methods described in previous sections.

### 6.1. Method of Lines in Space and Linearization

In this method, the derivative with respect to \( t \) is first discretized using a classical (iterative) \( \theta \)-method or a linearized \( \theta \)-method. This leads to an elliptic partial differential equation that must be solved at each time step. This can be done by using approximate factorization techniques and considering different values of \( \theta \) for the derivatives with respect to \( x \), i.e., \( \theta_x \), the derivatives with respect to \( y \), i.e., \( \theta_y \), and the reaction term, i.e., \( \theta_l \); interpolation may be used to get the solution in the whole domain.

Approximate factorization techniques can also be used with linearized \( \theta \)-methods. Since both \( \theta \) and linearized \( \theta \)-methods are similar, only the approximate factorization technique that uses the linearized \( \theta \)-method is described here.

The method of lines in space together with a fully linearized \( \theta \)-method for equation (77) results in

\[
(I + L_x + L_y)\Delta u + F(\Delta u) = G(u^n),
\]

where

\[
L_x \Delta u = \theta \Delta t \left( v_1 \frac{\partial}{\partial x} - D_1 \frac{\partial^2}{\partial x^2} \right) \Delta u,
\]

\[
L_y \Delta u = \theta \Delta t \left( v_2 \frac{\partial}{\partial y} - D_2 \frac{\partial^2}{\partial y^2} \right) \Delta u,
\]

\[
F(\Delta u) = \theta J f (u^n, x, y, t_n) \Delta u,
\]

\[
G(u^n) = f (u^n, x, y, t_n) + \theta T f (u^n, x, y, t_n) \Delta t.
\]

Different factorizations can be considered, e.g.,

\[
(I + L_x + \delta F)(I + L_y + \epsilon F), \quad E = \delta FL_y + \epsilon L_x F + L_x L_y + \epsilon \delta FF,
\]

\[
(I + L_y + \epsilon F)(I + L_x + \delta F), \quad E = \epsilon FL_x + \delta L_y F + L_y L_x + \epsilon \delta FF,
\]

\[
(I + L_x) (I + F) (I + L_y), \quad E = L_x F(I + L_y) + L_x L_y + FL_y,
\]

\[
(I + L_y) (I + F) (I + L_x), \quad E = L_y F(I + L_x) + L_y L_x + FL_x.
\]

where \( \epsilon + \delta = 1 \), and \( E \) is the factorization error. Note that, in general, \( L_x, L_y, \) and \( F \) do not commute. Moreover, the convection, diffusion, and reaction processes may be treated with different operators, so that, for example, the advection operator is solved by, say, the method of characteristics and its results are interpolated on a fixed grid which is the one used to solve
the diffusion operator; the reaction operator is, in this case, an ordinary differential equation which may be solved by means of standard techniques or the linearized methods presented in this paper. Furthermore, when solving multidimensional convection-reaction-diffusion equations, it is of paramount importance to take into consideration the different time and space scales which govern the physics of the problem, i.e., stiff chemical reactions and thin boundary layers, because the accuracy of an operator-splitting technique depends on the ratio between the largest and the smallest scales that govern the spatial and temporal evolution of the solution [32,45].

In this paper, the intermediate boundary conditions are those of the original problem. Moreover, it should be pointed out that if either standard $\theta$ or the linearized $\theta$-methods presented in this are employed to solve the advection-diffusion operators of equation (81), the discretization of the advective terms should be based on the absolute value of the mesh Reynolds/Péclet number; therefore, if upwind/donor approximations are used for the advection terms, then the spatial accuracy of the approximate factorization is $O(\Delta x)$. However, if the spatial discretization is performed with the C0 and C1 methods presented here, then the spatial accuracy of the discretization of advection-diffusion operators is second order provided that the spatial linearization is performed with respect to the midpoint of the subintervals employed by these methods.

It should also pointed out that, when a multidimensional problem is approximated by a sequence of one-dimensional ones, some uncoupling between the different directions is introduced in addition to other approximate factorization errors. Therefore, an accurate solution of multidimensional problems requires that the uncoupling between different spatial operators and the approximate factorization errors be minimized.

6.2. Methods of Lines in Time and Linearization

In this method, a grid of the closure of $\Omega$, with nodes $(x_i, y_j), i = 0, \ldots, N_x, and j = 0, \ldots, N_y,$ is considered. These nodes are used to discretize the spatial derivatives in equation (77), thus resulting in

$$\frac{dU}{dt} = (A_x + A_y)U + F(U, t),$$

(83)

where $U(t) = (u_{1,1}(t), u_{1,2}(t), \ldots, u_{N_x-1,N_y-1}(t))^T$, $u_{i,j}(t)$ is the approximate value of $u(x_i, y_j, t)$, and $F(u, t) = (f(u_{1,1}(t), x_1, y_1, t), \ldots, f(u_{N_x-1,N_y-1}(t), x_{N_x-1}, y_{N_y-1}, t))^T$. Note that the difference equations depend on the discretization of the partial derivatives of $u$ with respect to $x$ and $y$ which determine $A_x$ and $A_y$. The same operators, $\Delta^x$ and $\delta$, described before, may be used here.

Piecewise-linearized methods lead to

$$\frac{dU}{dt} = (A_x + A_y)U + F(U_n, t_n) + FT(U_n, t_n)(t - t_n) + JF(U_n, t_n)(U - U_n),$$

(84)

with $t \in [t_n, t_n+1]$. These methods yield piecewise continuous solutions in time in a finite set of points of $\Omega$.

It is also possible to use operator-splitting techniques as follows. First, the solution of equation (83) can be expressed as

$$U = U_p + V,$$

(85)

where

$$U_p = [F(U_n, t_n) - JF(U_n, t_n)U_n](t - t_n) + FT(U_n, t_n)(t - t_n)^2$$

(86)

is a particular solution of the nonhomogeneous problem, and $V$ is the general solution of the homogeneous problem

$$\frac{dV}{dt} = (A_x + A_y + JF(U_n, t_n))V.$$ 

(87)
This equation has a source term \( JF(U_n, t_n) \) that is necessary to consider in the factorization. One possibility is to solve sequentially
\[
\frac{dV}{dt} = (A_x + \delta JF(U_0, 0))V, \quad t \in [0, t_1], \quad V(0) = U_0,
\]
\[
\frac{dU}{dt} = (A_y + \epsilon JF(V(t_1), 0))U, \quad t \in [0, t_1], \quad U(0) = V(t_1),
\]
where \( \epsilon + \delta = 1 \), and to repeat the process in each time interval. As stated above, the physics of the problem, i.e., time and space scales, are of paramount importance in determining the optimum sequence of operators that result in the least error, and such considerations may result in a different splitting than that of equation (88) \[32,45\]. Moreover, the accuracy of methods of lines in time depends very much on the discretization of the spatial derivatives whose accuracy, in turn, depends on the mesh Reynolds number; therefore, if the advection terms are discretized by means of first-order accurate upwind differences which introduce artificial diffusion, even an accurate solution of equation (88) will suffer from spatial numerical errors.

Equation (83) can also be solved by using numerical methods which discretize the time variable, like the classical \( \theta \)-methods that lead to nonlinear schemes, or the linearized \( \theta \)-methods that result in linear ones. Depending on the type of linearization employed, different schemes can be obtained. Fully linearized \( \theta \)-methods yield a system of linear algebraic equations with a block tridiagonal matrix. Diagonally linearized \( \theta \)-methods result in a tridiagonal system for every component of \( u \). Triangular linearized \( \theta \)-methods lead to sequentially coupled tridiagonal systems for every component of \( u \). The order of the discretization error for these methods is \( O(\Delta t, \Delta x, \Delta y) \) if \( \theta \neq 0.5 \) and/or partial linearization is used, and \( O(\Delta t^2, \Delta x, \Delta y) \) if \( \theta = 0.5 \) and full linearization or iteration is used and the convection terms are approximated by first-order accurate finite difference discretizations.

All linearized \( \theta \)-methods lead to a system of algebraic equations like
\[
AU_{n+1} = BU^n + F(U^n) + \theta [FT(U^n) \Delta t + JF(U^n) \Delta U] + C_{n+1},
\] (89)
where \( U^n = (u^n_{1,1}, u^n_{1,2}, \ldots, u^n_{N_x-1,N_y-1})^\top \), \( F(U^n) = (F^n_{1,1}, F^n_{1,2}, \ldots, F^n_{N_x-1,N_y-1})^\top \), \( N_x \) and \( N_y \) are the number of points in the \( x \) and \( y \) directions, respectively, \( F^n_{i,j} = f(u^n_{i,j}, x_i, y_j, t_n) \), \( FT(U^n) \) is the vector of the partial derivatives of \( F \) with respect to \( t \), \( JF(U^n) \) is the Jacobian matrix of \( F \) with respect to \( U \), \( C \) depends on the boundary conditions, and \( A \) and \( B \) depend on the operators used to discretize the partial derivatives of \( u \) with respect to \( x \) and \( y \). These matrices can be expressed as
\[
\begin{bmatrix}
T_1 & M_1 & O & O & \cdots & O & O \\
N_2 & T_2 & M_2 & O & \cdots & O & O \\
O & N_3 & T_3 & M_3 & \cdots & O & O \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
O & O & O & O & \cdots & N_{N_y-1} & T_{N_y-1}
\end{bmatrix},
\] (90)
where, if the problem to be solved is governed by a single partial differential equation, \( T_i \) is a tridiagonal matrix, \( N_i \) and \( M_i \) are diagonal matrices, and \( O \) is a null matrix, all of which have dimensions equal to \( (N_x - 1) \times (N_x - 1) \); if the problem is governed by a system of partial differential equations, then \( T_i \) is a block tridiagonal matrix, and \( N_i \) and \( M_i \) are block diagonal matrices. It must be pointed out that it is possible to reduce \( A \) and \( B \) to block diagonal matrices with block dimensions equal to those for just one equation.

6.3. Factorization Techniques and Linearized \( \theta \)-Methods

The method of lines in space based on the fully linearized \( \theta \)-method, together with the discretization of the spatial derivatives using the \( \Delta^0 \) and \( \delta \) operators, leads to the following linear system:
\[
(I + L_x + L_y + L_f)\Delta u_{i,j} = R^n_{i,j},
\] (91)
where

\[ \Delta u_{i,j} = u_{i,j}^{n+1} - u_{i,j}^n, \]

\[ L_x = \Delta t \theta \left[ v_{1,j}^{n+1} \Delta x - D \delta^2_x \right], \]

\[ L_y = \Delta t \theta \left[ v_{2,i,j}^{n+1} \Delta y - D \delta^2_y \right], \]

\[ L_f = -\Delta t \theta f \left( u_{i,j}^n, x_i, y_j, t_n \right), \]

\[ R_{i,j}^n = \Delta t \left[ (D (\delta^2_x + \delta^2_y) + (\theta v_{1,j}^{n+1} - v_{1,i,j}^n) \Delta x + (\theta v_{2,j}^{n+1} - v_{2,i,j}^n) \Delta y) \right] u_{i,j}^n \]

\[ + f \left( u_{i,j}^n, x_i, y_j, t_n \right) + \theta T f \left( u_{i,j}^n, x_i, y_j, t_n \right) \]

Approximate factorization methods lead to, e.g.,

\[ (I + L_x - \delta L_f)(I + L_y - \epsilon L_f), \]

\[ (I + L_y - \epsilon L_f)(I + L_x - \delta L_f), \]

\[ (I + L_x)(I + L_f)(I + L_y), \]

\[ (I + L_y)(I + L_f)(I + L_x), \]

where \( \epsilon + \delta = 1; \) i.e., one partial differential equation results in two tridiagonal systems or two tridiagonal systems and a diagonal one depending on the factorization.

The factorization errors for equation (93) can be expressed as

\[ L_x L_y + \delta L_f L_y + \epsilon L_x L_f + \delta \epsilon L_f L_f, \]

\[ L_y L_x + \epsilon L_f L_x + \delta L_y L_f + \delta \epsilon L_f L_f, \]

\[ L_x L_f L_y + L_y L_f + L_f L_y, \]

\[ L_y L_f L_x + L_x L_f + L_f L_x, \]

respectively.

As stated above, the choice of an approximate factorization or operator-splitting technique depends on the physics of the problem being analyzed. For example, in air pollution problems which are characterized by stiff chemical reactions and large domains, the last factorization of equation (93) may be more accurate provided that the advection-diffusion operators are factorized into a sequence of advection and diffusion ones [32,45]; the advection operator may be solved by very accurate Riemann solvers or the method of characteristics.

For systems of partial differential equations, linearized \( \theta \)-methods lead to a system of equations with components of \( u \) coupled by the Jacobian matrix. Some modifications can be made to simplify the calculations, e.g., by using partially linearized \( \theta \)-methods, decomposition of the Jacobian matrix into the sum of an upper triangular matrix, a diagonal matrix, and a lower triangular matrix, i.e., \( L_f = L_U + L_D + L_L \), where \( L_U = -\Delta t \theta \text{upp}'(J f_{i,j}^n), \) \( L_D = -\Delta t \theta \text{diag}(J f_{i,j}^n), \) and \( L_C = -\Delta t \theta \text{low}'(J f_{i,j}^n), \) and \( \text{upp}'(A) \) and \( \text{low}'(A) \) denote the upper and lower matrices with zeros in their main diagonals and nonzero elements equal to those of \( A \), respectively. This decomposition allows the following approximate factorization:

\[ (I + L_x + \delta L_f)(I + L_y + L_D + \epsilon L_D), \]

where \( \epsilon + \delta = 1. \)

The approximate factorization errors can be eliminated by means of the following iterative predictor-corrector technique at each time level. The predictor scheme is given by

\[ (I + L_x + \delta L_f)\Delta u_{i,j}^{n,P} = R_{i,j}^n, \]

\[ (I + L_y + \epsilon L_f)\Delta u_{i,j}^{n,P} = \Delta u_{i,j}^{n,P}, \]
and the corrector one is

\[(I + L_x + \delta L_f)\Delta u^{n,k}_{i,j} = R^n_{i,j} - EF \left( \Delta u^{n,k-1}_{i,j} \right), \tag{98}\]
\[(I + L_y + \epsilon L_f)\Delta u^{n,k}_{i,j} = \Delta u^{n,k}_{i,j}, \tag{99}\]

where \(EF = L_xL_y + \delta L_fL_y + \epsilon L_xL_f + \epsilon\delta L_fL_f, \Delta u^{n,0}_{i,j} = \Delta u^{0}_{i,j},\) and \(k\) is the iteration.

This expression for \(EF\) involves nine grid-points and can be substituted by [46]

\[EF\Delta u^{n}_{i,j} = -L_x(\Delta u^{n}_{i,j} - \Delta u^{n-1}_{i,j}) + (\delta L_fL_y + \epsilon\delta L_fL_f)\Delta u^{n}_{i,j}, \tag{100}\]

which employs just five points.

The corrector step must be solved as many times as necessary until the following convergence criterion is satisfied:

\[\left\| \Delta u^{*n} - \Delta u^{*(n-1)} \right\| \leq \epsilon_c, \tag{101}\]

where \(\epsilon_c\) is a user-specified convergence tolerance.

This method is similar to the technique proposed by Steinthorsson and Shih [31] for a linear advection equation where it requires the elimination of a second-order spatial derivative, whereas the predictor-corrector method employed here is for reaction-diffusion equations and requires the elimination of a fourth-order derivative.

6.4. Application: A Two-Dimensional Reaction-Diffusion Problem

In order to illustrate the methods described in this section, the following problem is considered:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - u^2v, \\
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + u^2v - Kv, \quad (x, y) \in [-20, 20] \times [-20, 20],
\]

with initial and boundary conditions

- \(u(-20, y, t) = u(20, y, t) = u(x, -20, t) = u(x, 20, t) = 1,\)
- \(v(-20, y, t) = v(20, y, t) = v(x, -20, t) = v(x, 20, t) = 0,\)
- \(u(x, y, 0) = 1,\)
- \(v(x, y, 0) = \exp\left(-\left(x^2 + y^2\right)\right).\)

This problem is a generalization of the Twizell's one-dimensional problem analyzed in Section 5 but with a higher nonlinear reaction term.

The results shown here correspond to the different factorization techniques described in this section used together with linearized \(\theta\)-methods with \(\theta = 0.5\). Exact and approximate factorization techniques are compared, and the effect of different values \(\theta\) for the discretization of the partial derivatives with respect to \(x\) and \(y\) and for the reaction term is assessed.

Figure 9 displays the solution obtained with an exact factorization and the same parameters \((\theta_x = \theta_y = \theta_f = 0.5)\) for all the space derivatives, \(\Delta x = \Delta y = 0.5\) and \(\Delta t = 0.05\) at \(t = 0.8, 16,\) and 20 using a fully linearized \(\theta\)-method. It is observed that \(u\) evolves from a constant value 1 and decreases around \((0, 0)\) until a valley is formed, while \(v\) initially shows a peak near \((0, 0)\) that diffuses and decays until reaching an equilibrium state with four peaks near the corners.

Figure 10 illustrates the absolute errors of the above-described method but with the reaction term not equally distributed in the directions \(x\) and \(y\) at \(t = 8\) (\(\delta = 0\) and 1 in equation (93), top). It can be noted that the errors of both methods are similar in magnitude and trends, but
Figure 9. Solution of Twizell's two-dimensional reaction-diffusion problem at (from top to bottom) $t = 0, 8, 16, \text{ and } 20$. 
of opposite sign. The errors increase as time increases with maxima around the steepest regions of the solution.

Similar behaviour has been observed if different parameters $\theta$ are used for the reaction and diffusion terms [34]. Local errors are similar in magnitude to those shown in Figure 10 (bottom), even if the reaction term is not equally distributed in the $x$ and $y$ directions. In this case, the graphs are similar, but one is rotated 90 degrees with respect to the other.

Figure 11 shows the local difference between approximate and exact factorization techniques, at $t = 16$. It is observed that the differences between both methods are nearly three times greater for $u$ than for $v$.

Although not shown here, numerical experiments have also been performed with partially linearized methods and decomposition of the Jacobian matrix. These experiments show that diagonal linearization leads to greater errors than the other techniques considered here, while triangular linearizations yield similar results to those presented here.

If the Jacobian matrix is expressed as a sum of a diagonal, a lower triangular and an upper triangular one, and an approximate factorization technique is employed, the local errors are a bit larger than but exhibit similar trends to those shown in Figure 11.

7. CONCLUSIONS

Methods for solving ordinary and partial differential equation based on linearization techniques have been reviewed in this paper.

The piecewise-linearized technique for the solution of ordinary differential equations proposed here provides piecewise analytical approximations to initial-value problems, but at a high computational cost due to the need for evaluating matrix exponentials. Partial linearization can be used to avoid this drawback, although at the expense of a loss of accuracy.
Linearized $\theta$-methods for ordinary differential equations are based on the approximation of the nonlinear term of the classical $\theta$-scheme by its Taylor's first-degree polynomial. These methods are implicit for the same values as the classical ones and are A-stable, consistent, and convergent if a full linearization is considered; however, they may be computationally expensive, because they require the inversion of a matrix at each time step.

Partially linearized $\theta$-methods lead to uncoupled or sequentially coupled systems of algebraic equations which can be solved easily. Their accuracy is lower than that of fully linearized $\theta$-methods and depends on the order in which the equations are solved.

The fully and partially linearized $\theta$-methods presented here are Rosenbrock's and W-methods, respectively, although they have not been developed from linearly implicit Runge-Kutta techniques.

Piecewise-linearized methods and linearized $\theta$-methods presented here can be used together with a shooting technique to solve two-point boundary value problems in ordinary differential equations, although they may not converge. However, the piecewise linearization procedure allows for the development of new nonstandard exponential finite difference techniques that are not subject to the usual stability problems. These methods can be computationally expensive, but yield approximate analytical solutions.

Convection-reaction-diffusion problems can also be solved by means of linearized techniques together with methods of line in time or space and/or operator-splitting techniques. Factorization methods can be employed if multidimensional problems are considered.

Many numerical experiments have been performed by using piecewise-linearized and linearized $\theta$-methods for ODEs and reaction-diffusion problems, although only a few of them have been
shown here by conciseness. These experiments [34] indicate that the best accurate/computational cost ratio corresponds to methods of lines in time with triangularly linearized \( \theta \)-methods, and approximate factorization techniques for two- or three-dimensional problems.

The most stable methods described in this paper are methods of lines in space combined with the nonstandard finite difference methods C0 and C1, together with approximate factorization methods for multidimensional problems.

Partial differential equations with mixed derivatives can also be solved by using the techniques described in this paper, but they are currently under study.

REFERENCES