Classifying quadratic maps from plane to plane

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Abstract

Let \( V \) be a 2-dimensional vector space over an algebraically closed field \( \mathbb{F} \) of characteristic \( \neq 2, 3 \). The elements in \( S^2 V^* \otimes V \) under the natural action of \( \text{GL}(2, \mathbb{F}) \) are classified generically. A basis of invariants is also explicitly given.

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1. Introduction and statement of results

Let \( \mathbb{F} \) be a field of characteristic \( \neq 2, 3 \) and let \( V \) be a 2-dimensional \( \mathbb{F} \)-vector space. The goal of this paper is to classify the homogeneous quadratic maps \( q: V \rightarrow V \) that satisfy certain generic condition, called the “regularity condition” (see the formula (7)), under linear changes of coordinates in the case \( \mathbb{F} \) is algebraically closed.

If we take a basis \((v_1, v_2)\) in \( V \), then \( q(x) = q_1(x)v_1 + q_2(x)v_2 \) is a homogeneous quadratic map of the components \( x_1, x_2 \) of the vector \( x = x_1v_1 + x_2v_2 \); that is,

\[
q_1(x_1, x_2) = a_1 x_1^2 + 2b_1 x_1 x_2 + c_1 x_2^2, \quad q_2(x_1, x_2) = a_2 x_1^2 + 2b_2 x_1 x_2 + c_2 x_2^2,
\]

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dition problem in the subspace $F$ bilinear maps for scalars $a_i, b_i, c_i \in \mathbb{F}$, $i = 1, 2$. This class of functions plays an important role in several settings; notably, in the dynamics of real or complex discrete systems (e.g., see [2,5]); for example, if we take $\mathbb{F} = \mathbb{R}$, $\mathbb{V} = \mathbb{C}$, $v_1 = 1$, $v_2 = i$, the most representative quadratic function is, in complex notation, $x \mapsto x^2$, or with the same notations as above, $q(x) = (x_1^2 - x_2^2, 2x_1x_2)$.

Although $q$ can be described by a pair $(q_1, q_2)$ of quadratic forms—or even by a pair of symmetric $2 \times 2$ matrices as in the formula (3) below—the classification of homogeneous quadratic maps has nothing to do with that of bilinear forms or pencils (cf. [6–8]) due to the way on which the linear group acts on the former maps.

Let us explain this fact in detail. First of all, we recall that if $\text{char} \mathbb{F} \neq 2$, then there is a natural bijection between homogeneous quadratic maps on $V$ and symmetric bilinear maps $F: V \times V \rightarrow V$, which is given by the polarization formula $F(x, y) = (1/2)(q(x + y) - q(x) - q(y))$ and its inverse $q(x) = F(x, x)$ (e.g., see [4, XV, Sections 2–3]). If $V, W$ are two $\mathbb{F}$-vector spaces, we denote by $L(V, W)$ (resp. $L^2(V, W)$) the space of linear maps $F: V \rightarrow W$ (resp. bilinear maps $F: V \times V \rightarrow W$). We know that there are natural identifications, $L^2(V, V) \cong L(V \otimes V, V) \cong V^* \otimes V \otimes V = \otimes^2 V^* \otimes V$. Accordingly, our original goal becomes a classification problem in the subspace $S^2 V^* \otimes V \subset \otimes^2 V^* \otimes V$ of symmetric tensors of contravariant degree 1 and covariant degree 2 in the plane.

If $(\tilde{v}_1, \tilde{v}_2)$ is another basis in $V$, and $A \in \text{GL}(V)$ is the automorphism defined by $A(v_1) = \tilde{v}_1, A(v_2) = \tilde{v}_2$, then the expression of the quadratic map $q$ in the new basis is $x \mapsto A(qA^{-1}(x))$, as we need to express the vector $x$ as well as its image $q(x)$ in the new basis. Similarly, for the bilinear form $F$ corresponding to $q$, we have $(x, y) \mapsto A(F(A^{-1}(x), A^{-1}(y)))$. Therefore, the natural action of the linear group $\text{GL}(2, \mathbb{F})$ on $S^2 V^* \otimes V$ is given by

$$A \cdot F(x, y) = A(F(A^{-1}(x), A^{-1}(y))),$$

where $x, y \in V, F \in S^2 V^* \otimes V, A \in \text{GL}(2, \mathbb{F})$. Hence, two symmetric bilinear maps $F, G: V \times V \rightarrow V$ are $\text{GL}(2, \mathbb{F})$-equivalent if and only if there exists $A \in \text{GL}(2, \mathbb{F})$ such that

$$A(G(x, y)) = F(A(x), A(y)), \quad \forall x, y \in V.$$ 

Let $(v_1^*, v_2^*)$ be the dual basis of $(v_1, v_2)$; i.e., $v_i^* (v_j) = \delta_{ij}$. Every $F \in S^2 V^* \otimes V$ is written as

$$F = a_1 v_1^* \otimes v_1 + a_2 v_1^* \otimes v_2 + a_3 (v_1^* \otimes v_1^* + v_2^* \otimes v_1^* + v_1^* \otimes v_2^* + v_2^* \otimes v_2^*) \otimes v_1$$
$$+ b_2 (v_2^* \otimes v_2 + v_1^* \otimes v_1^* + v_1^* \otimes v_2^*) \otimes v_2 + c_1 v_2^* \otimes v_2 \otimes v_1 + c_2 v_2^* \otimes v_2 \otimes v_2.$$ 

Equivalently, $F$ can be described by two $2 \times 2$ symmetric matrices

$$F(x, y) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (x_1, x_2) \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

with $x = x_1 v_1 + x_2 v_2, y = y_1 v_1 + y_2 v_2$. 


Let us denote by $c_1^1$, $c_1^2$: $\otimes^2 V^* \otimes V \rightarrow V^*$ the contractions of the contravariant argument with the first and second covariant arguments respectively; precisely, $c_1^1(x^* \otimes y^* \otimes v) = x^*(v)y^*$, $c_1^2(x^* \otimes y^* \otimes v) = y^*(v)x^*$. The restriction of these two maps to the subspace of symmetric tensors, coincide; its common value is called the trace on $\otimes^2 V^* \otimes V$ and it is denoted by $\text{tr}$. From (2) we readily obtain

$$
\text{tr} F = (a_1 + b_2)v_1^1 + (b_1 + c_2)v_1^2.
$$

(4)

Hence the trace of a symmetric tensor $F \in \otimes^2 V^* \otimes V$ is a linear form $\text{tr} F: V \rightarrow \mathbb{F}$, and the map $F \mapsto \text{tr} F$ is proved to be $\text{GL}(2, \mathbb{F})$-equivariant. The name of trace for this linear form is justified by the following property. For a fixed $x \in V$, let $F_x: V \rightarrow V$ be the $\mathbb{F}$-linear endomorphism $F_x(y) = F(x, y)$, $\forall y \in V$. Then, we have $(\text{tr} F_x)(x) = \text{tr}(F_x)$, where, in the right hand side, tr has its usual meaning; that is, it is the trace of the endomorphism $F_x$.

Let $\sigma: V^* \rightarrow \otimes^2 V^* \otimes V$ be the map defined by

$$
\sigma(v^*)(x, y) = \frac{1}{2}(v^*(x)y + v^*(y)x), \quad x, y \in V, \quad v^* \in V^*.
$$

(5)

By using the formula (1), the homomorphism $\sigma$ is proved to be $\text{GL}(2, \mathbb{F})$-equivariant. We claim that, in addition, $\sigma$ is a section of the trace; i.e., $\text{tr} \circ \sigma$ is the identity map on $V^*$. In fact, if $v^* = \lambda_1 v_1^1 + \lambda_2 v_2^2$, then from the formula (5) we obtain

$$
\begin{align*}
\sigma(v^*)(v_1, v_1) &= \frac{1}{2}\lambda_1 v_1, \\
\sigma(v^*)(v_2, v_2) &= \frac{1}{2}\lambda_2 v_2, \\
\sigma(v^*)(v_1, v_2) &= \frac{1}{2}(\lambda_1 v_2 + \lambda_2 v_1)
\end{align*}
$$

and by using the formula (4) we conclude. This explains the role of the numerical factor $1/3$ in (5). There are other choices of $\sigma$ for which this factor is not required, but (5) seems to be more natural and, in any case, we are obliged to exclude characteristics 2 and 3 in order to apply Shephard–Todd’s theorem (see the proof of Lemma 6). Accordingly, we have a decomposition of $\text{GL}(2, \mathbb{F})$-modules $\otimes^2 V^* \otimes V = W \oplus \sigma(V^*)$, where $W = \{ F \in \otimes^2 V^* \otimes V: \text{tr} F = 0 \}$.

Each bilinear symmetric map $F: V \times V \rightarrow V$ can be assigned a quadratic form $Q_F: V \rightarrow \mathbb{F}$ defined by $Q_F(x) = \text{det}(F_x)$, where $F_x$ is the endomorphism defined above. As a simple computation shows, we have

$$
Q_F(x_1, x_2) = (x_1, x_2) \begin{pmatrix}
  a_1b_2 - a_2b_1 & \frac{1}{2}(a_1c_2 - a_2c_1) \\
  \frac{1}{2}(a_1c_2 - a_2c_1) & b_1c_2 - b_2c_1
\end{pmatrix} (x_1, x_2).
$$

(6)

For every $F \in \otimes^2 V^* \otimes V$ we set $\bar{F} = F - \sigma(\text{tr} F)$. Then, $F$ is said to be regular if the quadratic form $Q_{\bar{F}}$ is non-degenerate. A simple but rather long computation by using MapleV and (5), (4), (6) proves that $F$ is regular if and only if the following condition holds:
sider the following particularly simple symmetric bilinear map:

\[ F : \mathbb{R}^4 \rightarrow \mathbb{R} \]

Theorem 1. The group \( G \) of the quartic form (7) does not vanish. The first basic result is the following:

Lemma 2. The isotropy group of \( F \) is

\[ G = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}. \]

From the very definition it follows that the set of regular bilinear symmetric maps is an open subset \( R \subset S^2 V^* \otimes V \) in the Zariski topology; precisely, the set where the quartic form (7) does not vanish. The first basic result is the following:

Theorem 1. The group \( GL(2, \mathbb{F}) \) acts transitively on \( W \cap R \).

This result justifies the notion of regularity introduced above. In fact, \( GL(2, \mathbb{F}) \) does not act transitively on \( W \), as the rank of the quadratic form \( Q_F \) remains unchanged under the action of the linear group. (This is proved in the first paragraph of the proof of Theorem 1; see Section 2.1.) Hence we need to exclude the "degenerate part" in \( W \) in order to obtain a true orbit of the group.

Let a group \( G \) act on a set \( X \) on the left via a map \( G \times X \rightarrow X, (g, x) \mapsto g \cdot x \). The isotropy group of a point \( x \in X \) in \( G \)—usually denoted by \( G_x \), [4, I, Section 5]—is the subgroup of the elements \( g \in G \) keeping \( x \) fixed; that is, \( g \cdot x = x \). A natural bijection exists between the orbit \( G \cdot x \) and the quotient set \( G/G_x \).

The second step is to calculate the isotropy group of \( GL(2, \mathbb{F}) \) on \( W \cap R \). Consider the following particularly simple symmetric bilinear map:

\[ F_0(x, y) = \begin{pmatrix} (x_1, x_2) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, (x_1, x_2) \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \] (8)

Remark that \( F_0 \in W \cap R \). We have

\[ \det Q_F = \frac{4}{27} a_1 b_2 b_1 c_2 - \frac{1}{2} a_2 b_1 c_1 a_1 + \frac{2}{3} a_2 b_1 c_1 b_2 + \frac{1}{6} a_2 c_1 a_1 
- \frac{1}{3} a_2 c_1 b_2 - \frac{1}{27} a_1^3 c_1 + \frac{1}{27} a_1^3 b_1^2 + \frac{1}{108} a_1^3 c_2^2 
+ \frac{8}{27} b_2^2 c_1 + \frac{1}{27} b_2^2 b_1^2 + \frac{1}{27} b_2^2 c_2^2 + \frac{8}{27} a_2 b_1^2 - \frac{1}{27} a_2 c_2^2 
- \frac{8}{27} b_2^2 b_1 c_2 - \frac{1}{27} a_1^2 b_2 c_2 - \frac{4}{9} a_1 b_2^2 c_1 + \frac{2}{9} a_1^2 b_2 c_1 
+ \frac{7}{9} a_2 b_1^2 c_2 + \frac{4}{9} a_2 b_1^2 c_2 - \frac{1}{27} a_1 b_2 c_2^2 - \frac{2}{27} a_1^2 b_2^2 - \frac{1}{4} a_2^2 c_1^2 \neq 0. \] (7)

The proof of this lemma is deferred to Section 2.2.

According to Theorem 1 given \( F \in R \) there exists a matrix \( A \in GL(2, \mathbb{F}) \) such that \( \tilde{F} = F - \sigma(\text{tr } F) = A \cdot F_0 \). If \( B \in GL(2, \mathbb{F}) \) is another matrix such that \( \tilde{F} = F - \sigma(\text{tr } F) = B \cdot F_0 \), then \( B^{-1} A \in G \) and we have

\[ \text{tr}(B^{-1} \cdot F) = \text{tr}((B^{-1} A) \cdot (A^{-1} \cdot F)) = (B^{-1} A) \cdot \text{tr}(A^{-1} \cdot F). \]
Hence we have a well-defined quotient map
\[ \pi: R \to V^*/\mathcal{G}, \quad \pi(F) = \text{tr}(A^{-1} \cdot F) \mod \mathcal{G}, \]
where \( A \) is any matrix satisfying
\[ F = F = \sigma(\text{tr}F) = A \cdot F_0. \]  \hfill (10)

**Theorem 3.** The map \( \pi \) classifies the elements in \( R \); that is, two regular bilinear symmetric maps \( F, G \) are \( \mathcal{G} \)-equivalent if and only if \( \pi(F) = \pi(G) \).

**Proof.** Assume \( G = A \cdot F, A \in \text{GL}(2, \mathbb{F}). \) Then, from (10) we obtain
\[ \tilde{G} = G - \sigma(\text{tr}G) = A \cdot F - A \cdot \sigma(\text{tr}(F)) = A \cdot [F - \sigma(\text{tr}(F))] = (A)A \cdot F_0. \]

Hence
\begin{align*}
\pi(G) &= \text{tr}((AA^{-1}) \cdot G) \mod \mathcal{G} = \text{tr}(A^{-1} A^{-1}) \cdot G) \mod \mathcal{G} = \\
&= \text{tr}(A^{-1} A^{-1} \cdot (A \cdot F)) \mod \mathcal{G} = \text{tr}(A^{-1} \cdot F) \mod \mathcal{G} = \pi(F).
\end{align*}

Conversely, if \( \pi(F) = \pi(G) \), then there exists \( C \in \mathcal{G} \) such that \( \text{tr}H = 0 \), with \( H = CA^{-1} \cdot F - B^{-1} \cdot G \), where \( A \) is defined by (10) and \( B \) is defined by \( \tilde{G} = G - \sigma(\text{tr}G) = B \cdot F_0 \). From \( F = \sigma(\text{tr}F) + A \cdot F_0, G = \sigma(\text{tr}G) + B \cdot F_0 \) we have
\[ H = (CA^{-1} \cdot \sigma(\text{tr}F) + C \cdot F_0 - [B^{-1} \cdot \sigma(\text{tr}G) + F_0] \]
\[ = CA^{-1} \cdot \sigma(\text{tr}F) - B^{-1} \cdot \sigma(\text{tr}G) \]
\[ = \sigma(\text{tr}(CA^{-1} \cdot F)) - \sigma(\text{tr}(B^{-1} \cdot G)) \]
and taking the trace of \( H \) we conclude \( \text{tr}(CA^{-1} \cdot F) = \text{tr}(B^{-1} \cdot G) \), or equivalently \( \text{tr}G = \text{tr}(BCA^{-1} \cdot F) \). Hence, recalling that \( C \) belongs to the isotropy group,
\begin{align*}
G &= \sigma(\text{tr}G) + B \cdot F_0 = \sigma(\text{tr}(BCA^{-1} \cdot F)) + B \cdot F_0 \\
&= BC \cdot [\sigma(\text{tr}(A^{-1} \cdot F))] + F_0 = BC \cdot (A^{-1} \cdot F) \\
&= (BCA^{-1}) \cdot F. \qed
\end{align*}

**Theorem 4.** Let us consider the following polynomials on \( V^* \):
\[ f_1(x, y) = x^2 - xy + y^2, \quad f_2(x, y) = x^2 y - xy^2. \]
Then, we have

1. These polynomials are invariant under the action of \( \mathcal{G} \) on \( V^* \); hence they induce functions \( \bar{f}_i: V^*/\mathcal{G} \to \bar{V}, i = 1, 2 \).
2. Let \( I_i: R \to \bar{V} \) be \( I_i = \bar{f}_i \circ \pi, i = 1, 2 \). Two regular bilinear symmetric maps \( F, G \) are \( \text{GL}(2, \mathbb{F}) \)-equivalent if and only if \( I_i(F) = I_i(G), i = 1, 2 \). Therefore, \( I_1, I_2 \) classify the regular elements in \( S^2V^* \otimes V \).
The results in this paper seem to be of hard generalization to higher dimensions by using the same tools. First, because of the huge complexity of calculations when the dimension increases, but also due to the fact that GL(V) does not act transitively on \( W \cap R \) whenever \( \dim V > 2 \), and this is an essential point in deriving the results.

2. The generic orbit

2.1. Proof of Theorem 1

First of all, we remark that GL(2, \( F \)) really acts on \( W \cap R \). In fact, GL(2, \( F \)) transforms \( W \) into itself as \( W \) is a GL(2, \( F \))-module for the trace function is GL(2, \( F \))-equivariant. Let us prove that GL(2, \( F \)) also transforms regular elements in \( W \cap R \) into \( R \). If \( F \in W \cap R \), then \( A \cdot F = A \cdot F, \forall A \in \text{GL}(2, \( F \)) \) and according to (1) we obtain \((A \cdot F)_x = A \circ F_{A^{-1}(x)} \circ A^{-1}, x \in V \). Hence from the very definition of the quadratic form \( Q_F: V \rightarrow F \) we have \( Q_A \cdot F(x) = \det(A \cdot F)_x = \det F_{A^{-1}(x)} = Q_F(A^{-1}(x)) \); that is, \( Q_A \cdot F = Q_F \circ A^{-1} \). Therefore, \( Q_A \cdot F \) is non-degenerate if and only if \( Q_F \) is non-degenerate.

Consequently, it suffices to prove that for every \( F \in W \cap R \) there exist a basis \((v_1, v_2)\) of \( V \) and a matrix

\[
A = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \text{GL}(2, \( F \)),
\]

such that \( F = A \cdot F_0 \), where \( F_0 \) is the bilinear symmetric map defined in the formula (8) with respect to this basis. This is equivalent to saying \( A(F_0(x, y)) = F(A(x), A(y)), \forall x, y \in V \), or in the given basis,

\[
\begin{align*}
A(F_0(v_i, u_j)) &= F(A(v_i), A(u_j)), & i = 1, 2, \\
A(F_0(v_1, v_2)) &= F(A(v_1), A(v_2)).
\end{align*}
\]

Recall that \( F_0, F \) both are symmetric. By expanding this conditions and using the matrix notation in (3) we obtain

\[
\begin{align*}
a_1m^2 + 2b_1mp + c_1p^2 &= m, \\
a_2m^2 - 2a_1mp - b_1p^2 &= p, \\
a_1mn + b_1np + b_1mq + c_1pq &= m - n, \\
a_2mn - a_1np - a_1mq - b_1pq &= p - q, \\
a_1n^2 + 2b_1nq + c_1q^2 &= -n, \\
a_2n^2 - a_1nq - b_1q^2 &= -q.
\end{align*}
\] (11-13)

As \( F \) is traceless we have \( a_1 + b_2 = b_1 + c_2 = 0 \). Since \( Q_F \) is non-degenerate, there exists a basis \((v_1, v_2)\) of \( V \) such that \( Q_F(x) = 2x_1x_2, \) with \( x = x_1v_1 + x_2v_2 \). Hence from (6) we obtain
$Q_F(x) = -(a_1^2 + a_2 b_1) x_1^2 - (a_1 b_1 + a_2 c_1) x_1 x_2 + (a_1 c_1 - b_1^2) x_2^2 = 2x_1 x_2$

or equivalently,

(i) $a_1^2 + a_2 b_1 = 0$,  
(ii) $a_1 b_1 + a_2 c_1 = -2$,  
(iii) $a_1 c_1 - b_1^2 = 0$.  

If $a_1 \neq 0$, then from (14)-(iii) we obtain $c_1 = b_1^2/a_1$, and substituting it into (14)-(ii) we have $b_1 (a_1 + a_2 b_1/a_1) = -2$. Hence $b_1 \neq 0$. Then, from (14)-(i) we obtain $a_2 = -a_1^2/b_1$ and substituting $-a_1^2/b_1$ for $a_2$ into (14)-(ii) we are led to a contradiction. Hence $a_1$ must vanish and also $b_1$, as follows from (14)-(iii). Hence $b_2 = c_2 = 0$ by virtue of the traceless condition. Moreover, the equation (14)-(iii) now reads as $a_2 c_1 = -2$, thus implying $a_2 = 0$ and $c_1 = 0$.

Letting $a_1 = b_1 = b_2 = c_2 = 0$, $a_2 = -2/c_1$ in the equations (11)-(13), we obtain

(i) $c_1 p^2 = m$,  
(ii) $-2m^2 = c_1 p$.  

(i) $c_1 pq = m - n$,  
(ii) $-2mn = c_1 (p - q)$.  

(i) $c_1 q^2 = -n$,  
(ii) $2n^2 = c_1 q$.  

Eqs. (15) are equivalent to

(i) $m^3 = c_1/4$,  
(ii) $p = -2m^2/c_1$.  

Similarly, (17) are equivalent to

(i) $n^3 = -c_1/4$,  
(ii) $q = 2n^2/c_1$.  

As $F$ is algebraically closed and its characteristic $\neq 3$, the polynomial $X^3 - c_1/4$ has three simple roots: $\{m_1, m_2, m_3\}$. Each of them is a solution to (18)-(i), and the solutions to (19)-(i) are $\{-m_1, -m_2, -m_3\}$. This provides nine possible values for $(m, n)$, but we confine ourselves to the six values such that $m + n \neq 0$. Then, we have

$\det A = mq - np = \frac{2mn}{c_1} (m + n) \neq 0$.

Moreover, the equations (18)-(ii) and (19)-(ii) determine $p, q$ completely.

Hence we only need to prove that the equations (16) are a consequence of (18) and (19). To this end, we first remark that substituting (18)-(ii), (19)-(ii) into (16) we have

(i) $4m^2 n^2 = -c_1 (m - n)$,  
(ii) $mn = m^2 + n^2$.  

Set

$m = \sqrt[4]{\frac{4}{c_1} m}$,  
$n = \sqrt[4]{\frac{4}{c_1} n}$. 

Then (20) becomes

\[ P_1 \equiv \bar{m}^2 \bar{n}^2 + \bar{m} - \bar{n} = 0, \quad P_2 \equiv \bar{m}^2 + \bar{n}^2 - \bar{m}\bar{n} = 0 \]

and the resultant [4, IV, 8] of \( P_1, P_2 \) is

\[
\text{Res}(P_1, P_2) = \begin{vmatrix}
\bar{m}^2 & -1 & \bar{m} & 0 \\
0 & \bar{m}^2 & -1 & \bar{m} \\
1 & -\bar{m} & \bar{m}^2 & 0 \\
0 & 1 & -\bar{m} & \bar{m}^2
\end{vmatrix} = \bar{m}^2(\bar{m}^3 - 1)^2 = 0
\]

as \( \bar{m} \) is a root of \( X^3 - 1 \).

\[ \square \]

2.2. Calculating the isotropy of \( F_0 \)

For \( F = F_0 \) we have \( a_1 = 1, b_1 = 1, c_1 = 0, a_2 = 0, b_2 = -1, c_2 = -1 \), and substituting these values into the equations (11)–(13) we obtain the equations that define the isotropy group of \( F_0 \):

\[ (i) \quad m^2 + 2mp = m, \quad (ii) \quad p^2 + 2mp = -p, \quad (21) \]

\[ (i) \quad mn + np + mq = m - n, \quad (ii) \quad pq + np + mq = -p + q, \quad (22) \]

\[ (i) \quad n^2 + 2nq = -n, \quad (ii) \quad q^2 + 2nq = q. \quad (23) \]

First, assume \( m \neq 0 \). From (21)-(i) we obtain \( p = (1 - m)/2 \) and substituting \((1 - m)/2\) for \( p \) into (21)-(ii) we obtain \( m = 1, -1 \); hence \( p = 0, 1 \). If \( m = 0 \), then (21)-(i) holds identically and from (21)-(ii) we have \( p = 0, -1 \). Similarly, if \( n \neq 0 \), then from (23)-(i) we obtain \( q = -(n + 1)/2 \) and substituting \(-(n + 1)/2\) for \( q \) into (23)-(ii) we obtain \( n = 1, -1 \); hence \( q = -1, 0 \). If \( n = 0 \), then (23)-(i) holds identically and from (23)-(ii) we have \( q = 0, 1 \). Among all possible values found for \( (m, n, p, q) \) the equation (22) restricts those in (9) taking into account that \( \det A = mq - np \neq 0 \), thus concluding.

3. Proof of Theorem 4

We recall that a pseudoreflexion is a linear automorphism of \( V \) of finite order whose fixed points have codimension 1. A reflection is a diagonalizable pseudoreflexion of order 2 (see [1, 7.1]).

For the sake of simplicity we denote by \( \mathcal{I} \) the subring of \( \mathcal{G} \)-invariant polynomials; that is, \( \mathcal{I} = \mathbb{F}[x, y]^\mathcal{G} \). We know that \( \mathcal{I} \) is a graded \( \mathbb{F} \)-algebra: \( \mathcal{I} = \bigoplus_{i=0}^{\infty} \mathcal{I}^i \), where \( \mathcal{I}^i \) is the vector space of homogeneous polynomials of degree \( i \).
Lemma 5. We have

1. The isotropy group $\mathcal{G}$ is generated by reflections and contains exactly three pseudoreflections.

2. The polynomials $f_1$ and $f_2$ defined in Theorem 4 are $\mathcal{G}$-invariant.

3. $\mathcal{I}^0 = \mathcal{I}^1 = \{0\}, \mathcal{I}^2 = \mathbb{F} \cdot f_1, \mathcal{I}^3 = \mathbb{F} \cdot f_2.$

Proof. It is readily seen that

$$A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

are the only pseudoreflections in $\mathcal{G}.$ Moreover, as a simple computation shows, $A$ and $B$ generate the isotropy group; in fact, $\mathcal{G} = \{I, A, B, AB, BA, ABA\}.$ Accordingly, in order to prove that the polynomials $f_1$ and $f_2$ are $\mathcal{G}$-invariant we only need to check that they are invariant under $A$ and $B.$ This follows by simply using the equations of $A, B:$ $x^A = -y, y^A = -x; x^B = -x + y, y^B = y.$ As for the third item, it follows from a simple calculation.

Lemma 6. With the assumptions and hypotheses above, we have $\mathcal{I} = \mathbb{F}[f_1, f_2].$

Proof. As stated from the very beginning the characteristic of $\mathbb{F}$ is coprime to $|\mathcal{G}| = 6$ and so we can apply the Shephard–Todd theorem [1, Theorem 7.2.1]. By virtue of Lemma 5, $\mathcal{G}$ is generated by reflections and hence the aforementioned theorem warrants that there exist 2 = dim $V$ polynomials $g_1, g_2$ such that $\mathcal{I} = \mathbb{F}[g_1, g_2]$ is a polynomial ring, or, in other words, $g_1, g_2$ are two algebraically independent polynomials generating the ring of invariant polynomials. Furthermore, again from Shephard–Todd’s theorem we know that if $k_i = \deg g_i, i = 1, 2,$ then the number of pseudoreflections in $\mathcal{G}$ is $(k_1 - 1) + (k_2 - 1) = k_1 + k_2 - 2 = 3$ (item 1 in Lemma 5). As $\mathcal{I}^1 = \{0\}$ (item 3 in Lemma 5), we necessarily have $k_1 = 2, k_2 = 3,$ and we can conclude because $\mathcal{I}^2 = \mathbb{F} \cdot f_1, \mathcal{I}^3 = \mathbb{F} \cdot f_2.$

Therefore the item 1 in Theorem 4 directly follows from the item 2 in Lemma 5. As for item 2, we first remark that the functions in $\mathcal{I}$ separate $\mathcal{G}$-orbits (e.g., see [3, Theorem 5.52-ii]). Hence from Lemma 6 we conclude that if the covectors $w_i \in V^*$ satisfy $f_i(w_1) = f_i(w_2), i = 1, 2,$ then $w_1 \equiv w_2 \mod \mathcal{G}.$ Theorem 4 now follows from Theorem 3.

4. Computing the invariants

The goal of this section is to compute $I_i(F), i = 1, 2,$ explicitly in terms of the coefficients $a_i, b_i, c_i, i = 1, 2,$ of the matrices associated to $F$ (see the formula (3)).
We need to compute \( f_i(\text{tr}(A^{-1} \cdot F)) \), \( i = 1, 2 \), where

\[
A = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \text{GL}(2, \mathbb{F})
\]

is the matrix defined in (10). Hence, if we set

\[
\text{tr}(\tilde{F}(x, y)) = \begin{pmatrix} \tilde{a}_1 \tilde{b}_1 \tilde{c}_1 \\ \tilde{a}_2 \tilde{b}_2 \tilde{c}_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad (x_1, x_2) \begin{pmatrix} \tilde{a}_1 \tilde{b}_1 \tilde{c}_1 \\ \tilde{a}_2 \tilde{b}_2 \tilde{c}_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},
\]

then from the system (11)–(13) we deduce that the matrix \( A \) is defined by the following system:

\[
\begin{align*}
(i) & \quad \tilde{a}_1 m^2 + 2\tilde{b}_1 mp + \tilde{c}_1 p^2 = m, \\
(ii) & \quad \tilde{a}_2 m^2 - 2\tilde{a}_1 mp - \tilde{b}_1 p^2 = p, \\
(i) & \quad \tilde{a}_1 mn + \tilde{b}_1 np + \tilde{b}_1 mq + \tilde{c}_1 pq = m - n, \\
(ii) & \quad \tilde{a}_2 mn - \tilde{a}_1 np - \tilde{a}_1 mq - \tilde{b}_1 pq = p - q,
\end{align*}
\]

with

\[
\begin{align*}
\tilde{a}_1 &= \frac{1}{2}(a_1 - a_2), \quad \tilde{b}_1 = \frac{1}{2}(2b_1 - c_2), \quad \tilde{c}_1 = c_1, \\
\tilde{a}_2 &= a_2, \quad \tilde{b}_2 = \frac{1}{2}(2b_2 - a_1), \quad \tilde{c}_2 = \frac{1}{2}(c_2 - 2b_1)
\end{align*}
\]

as follows from the formulas (5) and (4), again recalling that \( \tilde{F} = F - \sigma(\text{tr } F) \).

As the trace function is \( \text{GL}(2, \mathbb{F}) \)-equivariant we have

\[
\text{tr}(A^{-1} \cdot F) = A^{-1} \cdot \text{tr } F = A \cdot ((a_1 + b_2)v_1^* + (b_1 + c_2)v_2^*),
\]

\[
= ((a_1 + b_2)m + (b_1 + c_2)p)v_1^* + ((a_1 + b_2)n + (b_1 + c_2)q)v_2^*.
\]

Hence, as a simple computation shows, we obtain

\[
I_1(F) = \chi_1(a_1 + b_2)^2 + \chi_2(a_1 + b_2)(b_1 + c_2) + \chi_3(b_1 + c_2)^2,
\]

\[
I_2(F) = \chi_4(a_1 + b_2)^3 + \chi_5(a_1 + b_2)^2(b_1 + c_2)
+ \chi_6(a_1 + b_2)(b_1 + c_2)^2 + \chi_7(b_1 + c_2)^3.
\]

with

\[
\begin{align*}
\chi_1 &= m^2 - mn + n^2, \\
\chi_2 &= (2p - q)m + (2q - p)n, \\
\chi_3 &= p^2 - pq + q^2, \\
\chi_4 &= m^2n - mn^2, \\
\chi_5 &= mq(m - 2n) - np(n - 2m), \\
\chi_6 &= 2pq(m - n) - mq^2 + np^2, \\
\chi_7 &= p^2q - pq^2.
\end{align*}
\]
Some extra calculus will also show that
\[-3X_1X_2^2X_4 + 4X_1^2X_2X_5 - 4X_1^3X_5 + 27X_3^2X_5 - 9X_4X_5^2 = 0. \tag{31}\]

Now the equations (24)–(26) can also be expressed in terms of the relationships (30), yielding:

\[
\begin{align*}
0 &= X_1 \bar{a}_1 + X_2 \bar{b}_1 + X_3 \bar{c}_1, \\
0 &= X_1 \bar{a}_2 - X_2 \bar{a}_1 - X_3 \bar{b}_1, \\
0 &= -2X_2 + 3X_3 \bar{a}_2 - X_5 \bar{a}_1 + X_6 \bar{b}_1 + 3X_7 \bar{c}_1, \\
0 &= -X_2 - X_4 \bar{a}_2 + 2X_5 \bar{a}_1 + 3X_6 \bar{b}_1 + 4X_7 \bar{c}_1, \\
0 &= -2X_3 + X_5 \bar{a}_2 - 2X_6 \bar{a}_1 - 3X_7 \bar{b}_1, \\
0 &= -2X_1 + 3X_4 \bar{a}_1 + 2X_5 \bar{b}_1 + X_6 \bar{c}_1.
\end{align*}
\]

This is a linear system that can be easily solved with the help of (31) for the unknowns $X_1, \ldots, X_7$, giving finally

\[
\begin{align*}
X_1 &= \frac{3(\bar{a}_1 \bar{c}_1 - \bar{b}_1^2)}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_2 &= \frac{3(\bar{a}_1 \bar{b}_1 + \bar{a}_2 \bar{c}_1)}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_3 &= \frac{-3(\bar{a}_1^2 + \bar{a}_2 \bar{b}_1)}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_4 &= \frac{\bar{c}_1}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_5 &= \frac{-3\bar{b}_1}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_6 &= \frac{3\bar{a}_1}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}, \\
X_7 &= \frac{\bar{a}_2}{6\bar{a}_1 \bar{a}_2 \bar{b}_1 \bar{c}_1 - 4\bar{a}_2 \bar{b}_1^3 - 3\bar{a}_1^3 \bar{b}_1^2 + 4\bar{a}_1^3 \bar{c}_1 + \bar{a}_2^2 \bar{c}_1}.
\end{align*}
\]

Substituting the values of $\bar{a}_i, \bar{b}_i, \bar{c}_i, i = 1, 2$, deduced from (27), into the equations above, and furtherly those into the equations (28) and (29), we finally have
\[
I_1(F) = \frac{1}{12 \det Q \mathcal{F}} \left\{ (a_1 + b_2)^2((2b_1 - c_2)^2 + 3(2b_2 - a_1)c_1) + (a_1 + b_2)(b_1 + c_2)((2b_2 - a_1)(2b_1 - c_2) - 9a_2c_1) + (b_1 + c_2)^2((2b_2 - a_1)^2 + 3(2b_1 - c_2)a_2) \right\}.
\]
\[ I_2(F) = \frac{1}{4} \det \overline{Q}_F \{-c_1(a_1 + b_2)^3 + (a_1 + b_2)^2(b_1 + c_2)(2b_1 - c_2) \\
+ (a_1 + b_2)(b_1 + c_2)^2(2b_2 - a_1) - a_2(b_1 + c_2)^3\}, \]

where \( \det \overline{Q}_F \) is defined in the formula (7).

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References