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STRONG REGULARITY AND GENERALIZED INVERSES IN JORDAN SYSTEMS
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Dedicated to the memory of Pere Menal

Abstract
A notion of generalized inverse extending that of Moore-Penrose inverse for continuous linear operators between Hilbert spaces and that of group inverse for elements of an associative algebra is defined in any Jordan triple system (J, P). An element \( a \in J \) has a (unique) generalized inverse if and only if it is strongly regular, i.e., \( a \in P(a)^2J \). A Jordan triple system \( J \) is strongly regular if and only if it is von Neumann regular and has no nonzero nilpotent elements. Generalized inverses have properties similar to those of the invertible elements in unital Jordan algebras. With a suitable notion of strong associativity, for a strongly regular element \( a \in J \) with generalized inverse \( b \) the subtriple generated by \( \{a, b\} \) is strongly associative.

1. Introduction.

An associative algebra \( A \) is strongly regular if \( a \in Aa^2, a \in A \). As it was shown in [2], strong regularity is a symmetric notion and can therefore be phrased in Jordan terms. Indeed, \( A \) is strongly regular if and only if \( a a^2Aa^2 \). Suppose now that \( (J, P) \) is a Jordan triple system (definition below) over an arbitrary ring of scalars \( \Phi \). An element \( a \in J \) is called strongly regular if

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We prove (Theorem 1) that \( a \in P(a)^2 J \) is strongly regular if and only if there exists a unique element \( b \in J \), called the generalized inverse of \( a \), satisfying \( P(a)b = a \), \( P(b)a = b \) and \( P(a)P(b) = P(b)P(a) \), and other equivalent conditions. This notion generalizes that of the Moore-Penrose inverse for continuous linear operators between Hilbert spaces and that of the group inverse for elements of an associative algebra (see [1]), and was already considered by Koecher in linear Jordan triple systems [12].

A classical result [7, p.261] asserts that an associative algebra \( A \) is strongly regular if and only if \( A \) is von Neumann regular and has no nonzero nilpotent elements. This result, which has been recently extended by Benslimane [2] to Jordan algebras, is improved here for Jordan triple systems. In fact we show (Theorem 3) that if \( J \) is a Jordan triple system without nonzero nilpotent elements (for instance a JB*-triple) then an element \( a \in J \) is strongly regular (equivalently \( a \) has a generalized inverse) if and only if some power of \( a \), say \( a^{2n+1} \) for \( n \geq 1 \), is von Neumann regular. As a consequence we get the following result of Kaplansky's [11]: Every element of a \(*\)-regular algebra \( A \) (\( A \) is a von Neumann regular associative algebra with an involution \( \ast \) such that \( x^\ast x = 0 \) implies \( x = 0 \)) has a Moore-Penrose inverse.

There exists a Jordan analogue to the abelian regular condition for associative algebras, which leads one to define abelian Jordan pairs and Jordan triple systems. A Jordan pair \( V \) is called abelian regular if it is regular and the Peirce-1-space \( V_1(e) = 0 \) for every (Jordan pair) idempotent \( e \). Just as in associative theory, a regular Jordan pair \( V \) is abelian regular if and only if \( V/P \) is a division Jordan pair for every prime ideal \( P \) of \( V \). However, the notions "strongly regular" and "abelian regular" which are the same for associative algebras [7, p.281], are quite different for Jordan triple systems. These results, which have been communicated to us by O. Loos, are also proved in this paper.

The analogy between classical inverses and generalized inverses is rather strong. We recall that a subalgebra \( S \) of a linear Jordan algebra \( J \) (1/2 in \( \Phi \)) is called strongly associative [23] if \((x.a)y = x.(a.y)\) for all \( x, y \in S \), \( a \in J \); equivalently, the subalgebra of \( \text{End}_\Phi(J) \) generated by the operators \( L_x \), \( x \in S \), where \( L_xa = x.a \) (\( a \in J \)), is commutative. It is not difficult to verify that \( S \) is strongly associative iff \( L(x,y) = L(y,x) \) for all \( x, y \in S \), where
\[ L(x, y)z = \{x \cdot y \cdot z\} = U_{x+z}y - U_x y - U_y z. \]

Suppose now that \( S \) is a subtriple of a JTS \( J \) over an arbitrary ring of scalars \( \Phi \). In general, the condition \( L(x, y) = L(y, x), \ x, y \in S \), where \( L(x, y)z = \{x \cdot y \cdot z\} = P(x+z)y - P(x)y - P(z)y \) is the polarization of the quadratic operator \( P \), does not imply the commutativity of the operators \( L(x, y), P(z), \ x, y, z \in S \), (if \( J \) has no 2-torsion then \( P(u) \) does commute with \( P(v) \) and \( L(u, v) \) with \( L(z, t) \) for all \( u, v, z, t \in S \), whenever \( L(x, y) = L(y, x) \) for all \( x, y \in S \). This leads to the following definition: a subtriple \( S \) of a JTS \( J \) is called strongly associative if \( L(x, y) = L(y, x) \) for \( x, y \in S \), and the subalgebra of \( \text{End}_\Phi(J) \) generated by \( L(x, y), P(z), x, y, z \in S \), is commutative. It is well known that the subalgebra generated by an element \( x \) in a linear Jordan algebra \( J \) is strongly associative; moreover, if \( x \) is invertible with inverse \( x^{-1} \) then the subalgebra generated by \( \{x, x^{-1}\} \) is also strongly associative. We prove (Theorem 4) that in any Jordan triple system \( J \) the subtriple generated by an element \( a \in J \), and the subtriple generated by \( a \) and its generalized inverse \( b \) (if this exists) are strongly associative.

Loos has implicitly considered the notion of generalized inverse in Jordan algebras. Lemma 1 of \([14]\) can be rephrased by saying that an element \( a \) in a Jordan algebra \( J \) has a generalized inverse iff there exists a (unique) idempotent \( e \in J \) such that \( a \) is invertible (in the usual sense) in the unital Jordan algebra \( U_e J \), with inverse \( a^{-1} \) (in \( U_e J \)) being the generalized inverse of \( a \).

In the last section we sketch how the classical notion of Drazin inverse can be studied in the setting of a Jordan triple system. We stress the relationship between Drazin inverse and Fitting decomposition in a Jordan triple system due to Loos \([14]\).

2. Preliminaries and notation.

A Jordan triple system (JTS) over an arbitrary ring of scalars \( \Phi \) is a \( \Phi \)-module \( J \) with a product \( P(x)y = P_x y \) quadratic in \( x \) and linear in \( y \), such that the identities:

\[ (\text{JTS1}) \quad L(P(x)y, y) = L(x, P(y)x) \]
hold in all scalar extensions, where \( L(x,y)z = \{x y z\} = P(x,z)y \) in terms of the polarization \( P(x,z)y = P(x+z)y - P(x)y - P(z)y \). Standard references for JTS are [13] and [18]. For the related notion of a Jordan pair the reader is referred to [13]. If \( J \) has no 6-torsion then the above conditions are equivalent to the following

\[
\text{(JTS4)} \quad \{x y z\} = \{z y x\}
\]

\[
\text{(JTS5)} \quad \{u v \{x y z\}\} - \{x y \{u v z\}\} = \{\{u v x\} y z\} - \{x \{v u y\} z\}
\]

In every JTS \( J \) the linear operator \( B(x,y) = \text{Id} - L(x,y) + P(x)P(y) \) satisfies the following identity of Macdonald type (JP26 of [13])

\[
\text{(2.1)} \quad P(B(x,y)z) = B(x,y)P(z)B(y,x).
\]

Other identities that we shall use along this paper are obtained by linearizing JTS1 and JTS2 (see p.14 of [13])

\[
\text{(2.2)} \quad L(x,y)P(x) = P(x, P(x)y) = P(x)L(y,x)
\]

\[
\text{(2.3)} \quad L(\{x y z\},y) = L(z,P(y)x) + L(x,P(y)z)
\]

\[
\text{(2.4)} \quad L(x,y)L(z,y) = P(x,z)P(y) + L(x,P(y)z)
\]

\[
\text{(2.5)} \quad L(x,y)P(x,z) = P(P(x)z,y) + P(x)L(y,z)
\]

\[
\text{(2.6)} \quad L(x,y)L(x,z) = L(P(x)y,z) + P(x)P(y,z)
\]

\[
\text{(2.7)} \quad L(x,\{y x z\}) = L(P(x)y,z) + L(P(x)z,y)
\]

By linearizing JTS3 we get

\[
\text{(2.8)} \quad P(P(x)z, P(x)y) = P(x)P(y,z)P(x).
\]
A unital Jordan algebra is a JTS $J$ with unit element $1$ ($P(1) = I_J$). Every (quadratic) Jordan algebra $J$ with products $Uxy$ and $x^2$ quadratic in $x$ and linear in $y$ gives rise to a JTS by defining $P(a) = U_a$. See [9] and [13] for basic facts about quadratic Jordan algebras; for (linear) Jordan algebras the reader is referred to [8] and [23]. Other examples of JTS are provided by associative triple systems.

A unital $\Phi$-module $B$ with a trilinear composition $<a b c>$ is called an associative triple system (ATS) of second kind if it satisfies:

\[(2.9) \quad <ab<cd>e> = <a<dcb>e> = <<abc>de>.
\]

Every ATS $B$ gives rise to a JTS $B^+$ by defining $P(a)b = <a b a>$, so we can carry over notions from JTS to ATS (basic notions on ATS are taken from [18]). Let $(J, P)$ be a JTS. Fixing an element $v \in J$, a squaring and a quadratic operator are defined by:

\[(2.10) \quad x^2 = x^{(2,v)} = P(x)v, \quad U_x = U^{(v)}(x) = P(x)P(v)
\]

with these operators $J$ becomes a Jordan algebra $J^{(v)}$ called the $v$-homotope of $J$. The (odd) powers of an element $x$ in a JTS are defined inductively by:

\[(2.11) \quad x^1 = x, \quad x^{2n+1} = P(x)x^{2n-1} \quad (n \geq 1),
\]

and $x$ is called nilpotent if $x^{2n+1} = 0$ for some $n$. We say that $J$ is anisotropic if $x^3 = 0$ implies $x = 0$. It is easily seen that this is the case if and only if $0$ is the only nilpotent element.

An element $a$ of a Jordan triple system $J$ is called von Neumann regular if $a = P(a)b$ for some $b \in J$. As it is well-known, we can arrange $P(b)a = b$ by replacing $b$ by $P(b)a$. Then we say that $(a,b)$ is an idempotent in the Jordan pair sense. An idempotent $(a,b)$ gives rise to two Peirce decompositions $J = J_2^+ \oplus J_1^+ \oplus J_0^+$ and $J = J_2^- \oplus J_1^- \oplus J_0^-$ where the first one is the associated to the orthogonal projections $E_2 = P(a)P(b)$, $E_1 = L(a,b)-2E_2$ and $E_0 = B(a,b)$ [13], and the second one is obtained by interchanging the roles of $a$ and $b$. The first Peirce decomposition coincides with the usual of the Jordan algebra $J(b)$ relative to the idempotent $a$, with $J_2^+ = J_2(b)(a), \quad J_1^+ = J_1(b)(a)$ and
Two idempotents \((a_1, b_1)\) and \((a_2, b_2)\) are associated if both of these Peirce decompositions agree.

3. Strong regularity.

3.1. Equivalent conditions for strong regularity. A Jordan semigroup is a nonempty set \(J\) with a map \(x \rightarrow P(x)\) of \(J\) into the monoid \(\text{Mon}(J)\) of maps of \(J\) into itself such that \(P(P(x)y) = P(x)P(y)P(x)\) (see [10] for related notions). Let \(J\) be a Jordan semigroup. An element \(a \in J\) is invertible if \(P(a)\) is bijective, and strongly regular if \(a \in P(a)^2 J\). Note that \([b] = P(b)J\) is a Jordan semigroup for every \(b \in J\). Strongly regular elements are clearly von Neumann regular \((a \in P(a) J)\) and have the following interesting characterizations.

**Theorem 1.** Let \(J\) be a Jordan semigroup and let \(a \in J\). Then the following conditions are equivalent:

1. \(a\) is strongly regular,
2. there exists \(b \in J\) such that \(P(a)b = a\) and \(P(a)b^3 = b\),
3. there exists \(b \in J\) such that \((a, b)\) is an idempotent and \(P(a)\) and \(P(b)\) commute,
4. \(a \in [a]\) and \(a\) is invertible in \([a]\).

If these conditions hold then:

i) \(b\) is uniquely determined by \(a\): \(b\) is the inverse of \(a\) in \([a]\),
ii) if \(a \neq 0\) then \(a\) is no nilpotent.

The element \(b\) will be called the *generalized inverse* of \(a\). Note that by symmetry of (3), \(b\) is strongly regular with generalized inverse \(a\).

**Proof.** (1) \(\Rightarrow\) (2). Let \(a = P(a)^2 x\) for some \(x \in J\) and take \(b = P(a)x\). Then \(P(a)b = a\) and \((P(a)P(b) = P(a)^2 P(x)P(a)^2 x = P(a)^2 x = P(a) x = b\).

(2) \(\Rightarrow\) (3). Let \(b \in J\) be such that \(P(a)b = a\) and \(P(a)b^3 = b\). Then \(P(b) = P(a)P(b)^3 P(a)\) and \(P(a) = P(a)P(b)P(a)\). Hence \(P(a)P(b)^2 = P(a)P(b)P(b) = P(a)P(b)P(a)P(b)^3 P(a) = P(a)P(a)^3 P(a) = P(b) = P(b)^2 P(a)\) by symmetry. Thus \(P(a)P(b) = P(a)P(b)^2 P(a) = P(b)P(a)\). Now \(P(b)a = P(b)P(a)b = P(a)P(b)b = b\).
If \((a, b)\) is an idempotent, \(a = P(a) b \in [a]\) and \(P(a) = P(a)P(b)P(a)\) so, by commutativity of \(P(a)\) and \(P(b)\), \(P(a)_{[a]}\) is bijective.

As \(a \in [a]\) and \(P(a)_{[a]}\) is in particular surjective, \(a = P(a)P(a)x\) for some \(x \in J\), so \(a\) is strongly regular.

Clearly every tripotent \(e\) \((P(e)e = e)\) is its own generalized inverse, and for an invertible element \(x\) in a Jordan monad \(J\) [10, Theorem of inverses] its inverse \(x^{-1}\) is the generalized inverse of \(x\). In the case of a Jordan triple system there are two more characterizations of strongly regular elements.

**Theorem 2.** Let \(J\) be a Jordan triple system and let \(a \in J\). Then \(a\) is strongly regular if and only if one of the following two conditions holds:

1. There exists \(b \in J\) such that \((a, b)\) is an idempotent associated to \((b, a)\),
2. \(a\) is regular and \(J = \text{Ker} P(a)_{[a]}\).

**Proof.** (5) \(\Rightarrow\) (6). Clearly \(a\) is regular. Considering the Peirce decomposition of \(J\) with respect to \((a, b)\), we have by [13, 5.4 (3)] that \(J = \text{Ker} P(b) \oplus [a]\), but \(\text{Ker} P(b) = \text{Ker} P(a)\) since \((a, b)\) and \((b, a)\) are associated. Now we use the equivalence (3) \(\Leftrightarrow\) (4) of Theorem 1.

(6) \(\Rightarrow\) (4). If (6) holds then \(P(a) : [a] \rightarrow [a]\) is surjective since \([a] = P(a)J = P(a)(\text{Ker} P(a) \oplus [a]) = P(a)[a]\). Clearly \(P(a)_{[a]}\) is injective whence \(P(a)_{[a]}\) is bijective.

(3) \(\Rightarrow\) (5) \(b = P(b)P(a)b = P(a)P(b)b\) implies that \(b \in P(a)J = P(b)J\) is invertible in the Jordan pair \((P(a)J, P(a)J)\) with inverse \(a\). Hence the idempotents \((a, b)\) and \((b, a)\) are associated by Lemma 1 of [22].

Condition (6) was communicated to us by the referee.

3.2. Jordan triple systems where all elements are strongly regular. The following theorem improves a well known result for associative algebras, [7,p.26], that was recently extended by Benslimane, [2], to Jordan algebras.

**Theorem 3.** Let \(J\) be a Jordan triple system.

(1) If \(J\) is anisotropic, then an element \(x \in J\) is strongly regular if and only if \(x^{2n+1}\) is von Neumann regular for some \(n \geq 1\).
The following conditions are equivalent:
(i) \( J \) is strongly regular
(ii) \( J \) is von Neumann regular and anisotropic.

\textbf{Proof.} (1). If \( x \) is strongly regular with generalized inverse \( v \), then
\[ P(x^3)v^3 = P(x)^2P(x)v^3 = P(x)P(x)v = P(x)x = x^3, \]
so \( x^3 \) is von Neumann regular. In fact \( x^{2n+1} = P(x^{2n+1})y^{2n+1} \) for all \( n \geq 1 \). Conversely, suppose that \( x^{2n+1} = P(x^{2n+1})y \) for some \( n \geq 1 \) and write \( u = x^{2n-1}P(x)^2ny = x^{2n-1}P(x^{2n-1})z \) with \( z = P(x)y \). By (JP23) of [13]
\[ P(x^{2n-1}P(x^{2n-1})z)u = B(x^{2n-1}, z)P(x^{2n-1})u. \]
Hence
\[ u^3 = P(u)u = P(x^{2n-1}P(x^{2n-1})z)u = B(x^{2n-1}, z)P(x^{2n-1})u = B(x^{2n-1}, z)P_x^{2n-2}(x^{2n+1}P(x^{2n+1})y) = 0, \]
so \( u = 0 \) and \( x^{2n+1} = P(x)^2ny = P(x^{2n-1})P(x)y \).
By repeating this process we get \( x = P(x)^{n+1}y \). Thus \( x \) is strongly regular.
(2). The implication (i) \( \Rightarrow \) (ii) follows from the last part of Theorem 1, and the implication (ii) \( \Rightarrow \) (i) is a direct consequence of (1).

Recently another characterization of strong regularity for Jordan triple systems has been given by Loos and Neher in [16, Corollary 6.5].

3.3. A strongly regular element generates a strongly associative subtriple.
Let \( J \) be a (linear) Jordan algebra over \( \Phi \) (1/2 \( \in \Phi \)). A subalgebra \( S \) of \( J \) is called strongly associative if \( x.(a.y) = (x.a)y \), \( x, y \in S, a \in J \) equivalently the subalgebra of \( \text{End}_\Phi (J) \) generated by the operators \( L_x \) \( (x \in S) \), where \( L_xa = xa \) \( (a \in J) \) is commutative. It follows from the identity

\[ (3.1) \quad L(x,y)z = 4L_yL_x(z) - \{y, z, x\} \]

that \( S \) is strongly associative if and only if \( L(x,y)L(y,x), x, y \in S, \) where \( L(x,y) = \{x, y, z\} = U_{x+y} - U_{x}y - U_{z}y. \) Suppose now that \( S \) is a subtriple of a JTS \( J \) over an arbitrary ring of scalars \( \Phi \). In general, the condition \( L(x,y) = L(y,x), x, y \in S, \) does not imply the commutativity of the operators \( L(x, y), P(z), x, y, z \in S. \) So a subtriple \( S \) of \( J \) will be called strongly associative if \( L(x,y) = L(y,x), x, y \in S, \) and the subalgebra of \( \text{End}_\Phi (J) \) generated by the
operators $L(x,y), P(z)$, $x, y, z \in S$ is commutative. Nevertheless, if $J$ has no 2-torsion then $P(u)$ does commute with $P(v)$ and $L(u,v)$ with $L(z,t)$, $u, v, z, t \in S$, whenever $L(x,y) = L(y,x)$ for all $x, y \in S$, which follows from (2.6) and (2.8) respectively.

**Lemma.** Let $a$ be a strongly regular element of a Jordan triple system $J$ with generalized inverse $b$. For an odd $n > 1$ we define the negative powers of $a$ by $a^{-n} := b^n$. Then we have for all $m, n, k \in 2\mathbb{Z} + 1$:

i) $P(a^m)a^n = a^{2m+n}$,

ii) $\{a^m a^n a^k\} = 2 a^{m+n+k}$,

iii) $P(a)P(a^m, a^n) = P(a^m, a^n)P(a)$,

iv) $L(a^m, a^n) = L(a, a^{m+n-1}) = L(a^n, a^m)$.

**Proof.** Let $m = 2m' + 1$, $n = 2n' + 1$,

(i) follows from JTS3 taking into account that $a^{m} := b^{-m}$.

(ii). By symmetry we may assume that $m \leq k$.

$\{a^m a^n a^k\} = P(a^m, a^k)a^n = (by (2.8)) P(a)^m P(a, a^k)P(a)^{m+n}a = (by [21; 3.10 ii]) P(a)^{2m'+n'}P(a, a^{k+m+1})a = (by [21; 3.9 iii]) 2 a^{m+n+k}$.

(iii). Let's assume that $m \leq n$.

$P(a)P(a^m, a^n) = (by (2.8)) P(a)^m P(a, a^{m+n+1})P(a)^m = (by [21; 3.10 iii]) P(a)^m P(a, a^{m+n+1})P(a)^{m+1} = (by (2.8) again) P(a^m, a^n)P(a)$.

(iv). We must verify that $L(a^n, a) = L(a, a^n)$ for all $n \in 2\mathbb{Z} + 1$. By [21; 3.11] if $n \geq 0$, the identity is satisfied. Then we have only to prove that $L(a,b^n) = L(b^n, a)$ for all $n \geq 0$. For $n = 1$ this is true since the idempotents $(a,b)$ and $(b, a)$ are associated by Theorem 2. For $n > 1$ we have $L(a, b^n) = L(a, P(b)b^{n-2}) = (by (2.3)) L((b^{n-2} b a), b) - L(b^{n-2}, P(b)a) = L(b^{n-2}, b)$ since $P(b)a = b$ and $\{b^{n-2} b a\} = L(a, b)b^{n-2} = (E_1 + 2 E_2) b^{n-2} = 2 b^{n-2}$, where $E_1, E_2$ denote the first Peirce projections relative to the idempotent $(a, b)$. Using (2.7) instead of (2.3) we can prove that $L(b^n, a) = L(b, b^{n-2}) = (by [21; 3.11]) L(b^{n-2}, b) = L(a, b^n)$.

Let's see that $L(a^m, a^n) = L(a, a^{m+n+1})$. If $m > 0$ and $n > 0$, we have the result by [21; 3.11]. If $m < 0$ and $n < 0$, we have the identity by [21; 3.11] again and considering that $L(b, b^{n-2}) = L(a, b^n)$. Let's suppose that $m > 0$ and $n < 0$. 


\[ L(a^m, a^n) = L(a^m, b^n) = L(P(a) a^{m-2}, b^n) = \text{(by (2.6)) } L(a, a^{m-2}) L(a, b^n) P(a) P(a^{m-2}, b^n) = \text{(by [21: 3.11] and (iii)) } L(a^{m-2}, a) L(b^n, a) P(a^{m-2}, b^n) P(a) = \text{(by (2.4)) } L(a^{m-2}, b^n) = \text{(reiterating the process) } L(a, b^{m-n+1}) = L(a, a^{m+n-1}). \]

**Theorem 4** Let \( J \) be a Jordan triple system over an arbitrary ring of scalars \( \Phi \).

1. For any element \( a \in J \) the subtriple generated by \( a \) is strongly associative.
2. If \( a \) is strongly regular with generalized inverse \( b \), then the subtriple generated by \( \{a, b\} \) is strongly associative.

**Proof.** (1). This can be found in H.P. Petersson's paper [21] by specializing his formulas (3.10)-(3.13) to the case where the Jordan pair is \( V^+=V^-=J \) and \( a=b \). Then Petersson's \( a_0^{m} \) is just our \( a^{2m-1} \). Note that by JTS3 and [21; 3.9ii], the subtriple generated by \( a \) is the linear span of \( \{a^n : n > 0, n \in 2 \mathbb{Z} + 1\} \).

(2). Suppose now that \( a \) is strongly regular with generalized inverse \( b \) and let \( \langle a, b \rangle \) denote the subtriple of \( J \) generated by \( \{a, b\} \).

(3.2) \( \langle a, b \rangle \) is the linear span of the set \( \{a^n : n \in 2 \mathbb{Z} + 1\} \).

We must only verify that this linear span is actually a subtriple, but this is a direct consequence of (i) and (ii) of the lemma.

Let \( \mathcal{M}(a, b) \) denote the subalgebra of \( \text{End}_\Phi(J) \) generated by \( \{1_J, P(x), L(y, z) : x, y, z \in \langle a, b \rangle \} \)

(3.3) \( \mathcal{M}(a, b) \) is generated by the operators \( 1_J, P(a), P(b), L(a, a^n) \) \( (n \in 2 \mathbb{Z} + 1) \).

By (3.2) and linearity we must only consider the operators of the form \( L(a^m, a^n) \) and \( P(a^m, a^n), m, n \in 2 \mathbb{Z} + 1 \). But \( L(a^m, a^n) = L(a, a^{m+n-1}) \) by (iv) of the lemma. Now for \( m < n \) and \( m = 2m'-1 \), we have \( P(a^m, a^n) = P(a)^{m'} P(a, a^{n-m+1}) P(a)^{m'} = \text{(by (2.2)) } P(a)^{m'} L(a, a^{n-m+1}) P(a)^{m'+1} \).

(3.4) \( \mathcal{M}(a, b) \) is commutative.
By (3.3) we must only verify the commutativity of the operators of the form \( P(a), P(b), L(a, P(a)^2a) \). By Theorem 1 the operators \( P(a) \) and \( P(b) \) commute. Now we have

(i) \( P(a)L(a, a^n) = (\text{by (iv) of the lemma}) P(a)L(a^n, a) = (\text{by (2.2)}) L(a, a^n)P(a) \).

(ii) \( P(b)L(a, a^n) = (\text{by (iv) again}) P(b)L(a^{-1}, a^{n+2}) = (\text{by (iv)}) P(b)L(a^{n+2}, b) = (\text{by (2.2)}) L(b, a^{n+2}) P(b)= (\text{by (iv)}) L(a, a^n)P(b) \).

(iii) Let \( [A, B] = AB - BA \) denote the commutator of \( A, B \).

\[ [L(a, a^n), L(a, a^m)] = (\text{by JTS5}) L([a a^n a, a^m]) - L(a, \{a^n a a^m\}) = \]

(by (ii) of the lemma) \( L(2a^{n+2}, a^n) - L(a, 2a^{n+m+1}) = 0 \) (by (iv)).

From (iv) of the lemma and (3.4) we get that the subtriple \( \langle a, b \rangle \) is strongly associative, as required.

Recall that a subtriple \( S \) of a JTS \( J \) is called flat if \( \{x y z\} = \{y x z\} \) for all \( x, y, z \in S \). Clearly, flatness is a weaker condition than strong associativity, so our theorem improves the well known result that the subtriple generated by an element is flat [19, p.150]. We will now study strongly regular elements in specific types of Jordan triple systems.

3.4. Characterization of strongly regular elements in Jordan triple systems without 2-torsion. Generalized inverses in linear Jordan triple systems have been considered by Koecher [12]. In fact condition (iii) of the next theorem is due to him.

**Theorem 5.** Suppose that \( J \) is a JTS without 2-torsion \( (2x = 0 \Rightarrow x = 0, x \in J) \). For \( a, b \in J \) the following conditions are equivalent:

(i) \( \{ a b b \} = 2b, P(a)b = a, \)

(ii) \( b \) is the generalized inverse of \( a, \)

(iii) \( P(a)b = a, P(b)a = b, L(a, b) = L(b, a). \)

**Proof.** (i) \( \Rightarrow \) (ii). \( 2b = \{ a b b \} \) implies by (2.5)

\[ 4b = 2L(a, b)b = L(a, b)P(a, b)b = P(P(a)b, b)b + P(a)L(b, b)b. \]

Hence, if \( P(a)b = a \) then

\[ 4b = \{ a b b \} + 2P(a)P(b)b \Rightarrow 2b = 2P(a)P(b)b \Rightarrow b = P(a)P(b)b. \]

Now (ii) \( \Rightarrow \) (iii) follows from Theorem 1(4), and (iii) \( \Rightarrow \) (i) is clear.
3.5. Characterization of strongly regular elements in Jordan algebras. As a particular case of Theorem 5, we have that an element $b$ in a linear Jordan algebra $J$ is strongly regular with generalized inverse $c$ if and only if:

$$c^2b = c \text{ and } b^2c = b.$$ 

Even for a Jordan algebra $J = A^+$, where $A$ is a unital associative algebra over a field of characteristic $\neq 2$, conditions $b^2c = b$ and $c^2b = c$ does not imply that $c$ is the generalized inverse of $b$, even if $b$ is invertible. Indeed, let $A = M_2(\mathbb{C})$ and consider $b = E_{11} + iE_{22}$, $c = E_{11} + \alpha E_{12} + \beta E_{21} - iE_{22}$, where $(E_{11}, E_{12}, E_{21}, E_{22})$ denotes the canonical basis of $M_2(\mathbb{C})$. We have that $b^2c = b$ and $c^2b = c$ whenever $\alpha \beta = 0$. However, the generalized inverse of $b$ is obtained by taking $\alpha = 0 = \beta$, i.e., $c = b^{-1}$.

The following result, that is implicitly proved in Lemma 1 of [14], provides an interesting local characterization of generalized inverses.

**Theorem 6.** Let $J$ be a Jordan algebra. An element $b \in J$ is strongly regular with generalized inverse $c$ if and only if there exists a unique idempotent $e \in J$ such that $b$ is invertible (in the usual sense) with inverse $b^{-1} = c$ in the unital Jordan algebra $U_0J$.

3.6. Characterization of strongly regular elements in associative triple systems. Let $H_1$, $H_2$ be two complex Hilbert spaces. We recall [1] that for a continuous linear operator $a : H_1 \to H_2$ the following conditions are equivalent:

1. the range of $a$ is closed,
2. there exists a unique continuous linear operator $b : H_1 \to H_2$, called the *Moore-Penrose inverse* of $a$, [20], satisfying the following conditions:

$$aa^*b = a = ba^*a, \quad bb^*a = b = ab^*b$$

where $a^*$ denotes the adjoint of $a$. Write $BL(H_1, H_2)$ to denote the Banach space of all continuous linear operators from $H_1$ to $H_2$. Then $BL(H_1, H_2)$ is an ATS of second kind under the triple product $<a \ b \ c> = ab^*c$. More generally, it makes sense (cf.[5]) to consider the notion of Moore-Penrose inverse in any ATS of second kind.
Proposition 1. Let B be an associative triple system of second kind. For a, b ∈ B the following conditions are equivalent:

(i) b is the Moore-Penrose inverse of a, i.e.,
\[ <a a b> = a = <b a a>, \quad <b b a> = b = <a b b> \]

(ii) b is the generalized inverse of a in the JTS B+.

Proof. (i) ⇒ (ii). Let b be the Moore-Penrose inverse of a. Then
\[ a = <a a b> = <a <a a b > b > = <a b <a a b >> = <a b a > = P(a)b \]
and
\[ P(a)P(b)b = <a <b b b > a > = <<a b b > b a > = <b b a > = b. \]
Hence b is the generalized inverse of a by Theorem 1(2).

(ii) ⇒ (i). Conversely, if \[ a = P(a)b = <a b a >, b = P(a)b^3 = <a <b b b > a > \]
then \[ <a b b > = <a b <a b b b > a > = <<a b a > <b b b > a > = <a <b b b > a > = P(a)b^3 = b. \]
Similarly \[ b = <b b a >. \]
The remaining conditions follow in virtue of the symmetry of Theorem 1(3).

Note that the proof of Proposition 1 does not require a module structure, but only identity (2.10).

3.7. Characterization of strongly regular elements in semigroups. Let a be a linear operator on a vector space V over a field K. A linear operator \[ a ∈ \text{End}_K(V) \]
is called the group inverse of a if the following conditions hold:

\[ (3.6) \quad aba = a, bab = b, \quad ab = ba. \]

It is well known (see [1]) that a has a (unique) group inverse if and only if \[ V = \text{Ker}(a) ⊕ \text{Im}(a). \]
The notion of group inverse can be defined in any semigroup [4].

Proposition 2. For a, b elements in a semigroup S the following conditions are equivalent:

(i) b is the group-inverse of a, i.e,
\[ aba = a, bab = b \] and \[ ab = ba \]

(ii) b is the generalized inverse of a in the Jordan semigroup S+

Proof. (i) ⇒ (ii) \[ P(a)P(b)b = ab^3a = b(aba)b = bab = b. \]

(ii) ⇒ (i) \[ aba = a \] and \[ ab^3a = b \] imply \[ b^2a = b(ba) = (ab^3a)ba = \]
ab^3(ab^2a) = ab^3a = b = ab^2 by symmetry. Hence ba = ab^2a = ab. Now bab = b^2a = b.

An associative algebra $A$ with involution $^* : A \to A$ is said to be $^*\text{-regular}$ [11] if $A$ is von Neumann regular and $xx^* = 0 \Rightarrow x = 0$. It is easy to see that an associative algebra $A$ with involution $^*$ is $^*\text{-regular}$ if and only if the JTS defined on $A$ by $P(x)y = xy^*x$ is von Neumann regular and has no nonzero nilpotent elements.

Corollary 1 (see [7] and [11]). Let $A$ be an associative algebra.

(1) $A$ is strongly regular if and only if $A$ is von Neumann regular with 0 as unique nilpotent element.

(2) If $A$ has an involution $^*$ then every element of $A$ has a Moore-Penrose inverse if and only if $A$ is $^*\text{-regular}$.

Let $J$ be a vector space over a field $K$ and $q : J \to K$ a quadratic form. Consider the corresponding Jordan triple system given by $P(x)y = q(x, y)x - q(x)y$, where $q(x, y) = q(x+y) - q(x) - q(y)$ is the polarization of $q$.

Proposition 3. Let $J$ be the Jordan triple system of a quadratic form $q$ and let $0 \neq a \in J$. Then the following conditions are equivalent:
(i) $a$ is strongly regular,
(ii) $q(a) \neq 0$,
(iii) $a^3 \neq 0$.
If these conditions are fulfilled then $q(a)^{-1}a$ is the generalized inverse of $a$.

Proof. Since $a^3 = q(a)a$, obviously (ii) $\Leftrightarrow$ (iii). If $a$ is strongly regular then $a^3 \neq 0$ by Theorem 1 whence (i) $\Rightarrow$ (iii). If $q(a) \neq 0$ then $b = q(a)^{-1}a$ satisfies condition (2) of Theorem 1.

3.9. Characterization of strongly regular elements in degree 3 algebras.
Consider now the problem of computing generalized inverses in Jordan algebras.
defined by cubic forms. First we recall the following Springer's theorem: Let 
(N, 1) be an admissible cubic form with base point on a vector space J over a
field K (see [17] for definition and notation). Define a quadratic operator
U:J→EndK(J) by Uab = T(a, b)1 - a#x b. Then (J, U, 1) is a Jordan algebra
and for any b ∈ J we have the Hamilton-Cayley theorem: b3 - T(b)b2 +
T(b#)b - N(b)1 = 0 [17; (20)].

Proposition 4. Let J be a Jordan algebra defined by an admissible cubic
form N. Then an element b ∈ J is strongly regular if and only if b3 ≠ 0 and
(b# = 0 or (b#)3 ≠ 0).

Proof. Since (Ux,y)# = Ux,y#, [17; (2.2)], if c is the generalized
inverse of b then c# is the generalized inverse of b#. Hence the condition is
necessary by Theorem 1. Conversely, suppose that b3 ≠ 0 and (b# = 0 or
(b#)3 ≠ 0). From Cayley-Hamilton theorem we have one of the following
possibilities:
(1) N(b) ≠ 0. Then b is actually invertible with b⁻¹ = N(b)⁻¹b#.
(2) N(b) = 0 and b# = 0. Then T(b) ≠ 0 and c = T(b)⁻²b is the
generalized inverse of b.
(3) N(b) = 0 and b# ≠ 0. Then T(b#) ≠ 0 because b#b = N(b)b = 0 (see
[17; Theorem 1(i)]) and N(b#) = 0. In this case c = b x T(b#)⁻²b# is the
generalized inverse of b.

4. Related notions.

There exists another regularity condition in associative algebras which is
equivalent to strong regularity, namely: the condition "abelian regular". An
associative algebra is called abelian regular if it is regular and every
idempotent is central [7]. O. Loos has communicated to us that abelian regular
condition has a Jordan analogue.

Lemma (Loos). Let A be an associative algebra and let V be the Jordan pair
(A, A) associated to A (with quadratic maps P(x)y = xyx). If e = (x, y) is an
idempotent of the Jordan pair V, then the idempotents xy and yx of the
algebra $A$ are central if and only if the Peirce-1-space $V_1(e) = 0$. In fact, $xy = yx$ in this case.

**Proof.** Suppose first that $xy$ and $yx$ are central. Then $axy + xya = a \Rightarrow ayxy + xyay = ay \Rightarrow 2ay = ay \Rightarrow ay = 0$. Similarly $ya = 0$ and hence $a = 0$. Since $x$ and $y$ play symmetrical roles we get that $V_1(e) = 0$. Conversely, if $V_1(e) = 0$ then the Peirce-1-space of the $y$-homotope of $A^+$ relative to the idempotent $x$ is equal to 0. Hence $x$ is a central idempotent of the associative algebra $A(y)$, i.e., $xya = ayx$ for any $a \in A$, but $(xy - yx)yx + xy(xy - yx) = xy^2x - yx + xy - xy^2x = 0 \Rightarrow xy = yx$. So $xya = ayx = axy$, which proves that $xy = yx$ is a central idempotent of $A$.

The above lemma suggests to call a Jordan pair $V$ abelian regular if $V$ is regular, and $V_1(e) = 0$ for every idempotent $e$ of $V$.

Let $V = (V^+, V^-)$ a Jordan pair. we say that $V$ is prime if $P^e (l_1^e, l_2^e) = 0$ ($e = \pm$) for $l_1, l_2$ ideals of $V$, implies $l_1 = 0$ or $l_2 = 0$. An ideal $l$ of $V$ is prime if the Jordan pair $V/I$ is prime. If $V$ is a nondegenerate Jordan pair (in particular if $V$ is regular), then it is prime if and only if $l_1 \cap l_2 = 0 \Rightarrow l_1 = 0$ or $l_2 = 0$, for $l_1, l_2$ ideals of $V$. Just in associative theory we have:

**Theorem 7** (Loos). A regular Jordan pair $V$ is abelian regular if and only if $V/P$ is a division pair, for every prime ideal $P$ of $V$.

**Proof.** Suppose first that $V = (V^+, V^-)$ is abelian regular and let $I = (I^+, I^-)$ a prime ideal of $V$. Given $x \in V^+$ such that $x \notin I$ choose $y \in V^-$ such that $e = (x, y)$ is an idempotent of $V$. Relative to this idempotent $V = V_2 \oplus V_0$ which implies that $V/I = (V/I)_2 \oplus (V/I)_0$ and hence, by Peirce relations, $(V/I)_2$, $(V/I)_0$ are ideals of $V/I$. Since $(V/I)_2 \neq 0$ we have that $(V/I)_0 = 0$ because $I$ is prime. So $V/I = (V/I)_2$ which proves that $V/I$ is a division Jordan pair. Suppose conversely that $V/I$ is a division Jordan pair for all prime ideal $I$ of $V$, and let $e$ be an idempotent of $V$. Assume that $0 \neq x \in V_1^+(e)$. By regularity, $x = f_+$ for a Jordan pair idempotent $f = (f_+, f_-) \in V_1(e)$. By Zorn's Lemma, there exists an ideal $l$ of $V$ which is maximal with respect to the property "$f_+ \notin I^+$ or $f_- \notin I^-". Then $l$ is prime: If $B$ and $C$ are ideals
of $V$ properly containing $1$ then $f_\epsilon \in \mathcal{B}_\epsilon \cap \mathcal{C}_\epsilon$ and thus $f_\epsilon = P(f_\epsilon)f_\epsilon \in P(\mathcal{B}_\epsilon)\mathcal{C}_\epsilon$, showing $P(\mathcal{B})\mathcal{C}$ is not contained in $1$. In $V = V/I$ we now have $0 \neq f \in V_1(e)$ which is a contradiction since $V$ is a division pair.

So it turns out that the notions "strongly regular" and "abelian regular" which are the same for associative algebras are quite different for Jordan triple systems (call a Jordan triple system abelian regular if the associated Jordan pair $(J, J)$ has this property). Clearly, a strongly regular Jordan triple system need not be abelian regular since, e.g., the Jordan triple system of all $n$-by-$n$ hermitian matrices over $\mathbb{C}$ is strongly regular but has lots of tripotents $e$ with $J_1(e) \neq 0$. But conversely, an abelian regular Jordan triple system is strongly regular by Theorem 3 because any nilpotent element would be zero modulo any prime ideal, and hence would be zero.

5. Drazin inverse in Jordan triple systems.

Drazin developed in [4] a notion of pseudo-inversibility slightly more general than that of group inverse and applicable to a large class of associative rings. In this section we show that the Drazin inverse is also a Jordan notion closely related to Fitting decomposition in Jordan triple system studied by Loos [14].

Let $A$ be an associative algebra. Recall that an element $a \in A$ is called strongly $\pi$-regular if there exist a positive integer $m$ and $b, c \in A$ such that:

\[(5.1) \quad ba^{m+1} = a^m = a^{m+1}c.\]

It is shown in [4] that $a \in A$ is strongly $\pi$-regular if and only if there exists a (unique) $x$ in $A$ called the Drazin inverse of $a$ satisfying:

\[(5.2) \quad a^n = a^{n+1}x, \quad ax = xa, \quad x = xax\]

for a certain positive integer $n$. It is not difficult to verify that both notions "strongly $\pi$-regular" and "Drazin inverse" have Jordan analogues, namely: $a \in A$ is strongly $\pi$-regular if and only if the sequence $P(a)A \supset P(a)^2A \supset \ldots$ stabilizes. In this case, $x$ is the Drazin inverse of $a$ if and only if:
for a certain \( n \geq 0 \), where \( P(b)c = bcb \).

Suppose now that \( J \) is an arbitrary Jordan triple system. An element \( a \in J \) will be called strongly \( \pi \)-regular if the sequence \( P(a)J \supset P(a)J \supset \ldots \) becomes stationary, but this is precisely a necessary and sufficient condition for the element \( a \) to have a Fitting decomposition in the sense of Loos. Then Theorem 3 of \([14]\) can be rephrased as follows:

**Theorem 8.** Let \( J \) be a Jordan triple system. For an element \( a \in J \) the following conditions are equivalent:

(i) \( a \) is strongly \( \pi \)-regular;

(ii) there exists a Peirce decomposition \( J = J_2 \oplus J_1 \oplus J_0 \) (necessarily unique) such that: \( a = a_2 + a_0 \in J_2 \oplus J_0 \), where \( a_2 \) is invertible in \( J_2 \) and \( a_0 \) is nilpotent. In this case, \( P(a)^n J = J_2 \) and \( \ker P(a)^n = J_1 \oplus J_0 \) for all sufficiently large \( n \);

(iii) there exists \( x \in J \) such that:

\[
P(a)^{2n+1} x^{2n+1} = a^{2n+1}, \quad P(a)P(x) = P(x)P(a), \quad P(x)a = x.
\]

If these conditions hold then \( x \) is unique and is called the Drazin inverse of \( a \).

**Sketch of the proof.** (i) \( \Rightarrow \) (ii) has been proved in \([14, \text{Theorem 3}]\).

(ii) \( \Rightarrow \) (iii). It follows from the properties of Fitting decomposition and that \( a_2 \) is strongly regular with generalized inverse \( x \).

(iii) \( \Rightarrow \) (i). By Theorem 1 \( a^{2n+1} \) is strongly regular, so, \( a^{2n+1} \in P(a^{2n+1})J \) implies that the sequence \( P(a)J \supset P(a)J \supset \ldots \) becomes stationary.

**Corollary 3.** Suppose that \( J \) satisfies descending chain condition on principal inner ideals (for instance, \( J \) is a nondegenerate Jordan triple system which coincides with its socle, \([6, [15]]\)). Then every element of \( J \) has a Drazin inverse.

**Remark.** Consider two complex rectangular matrices \( B \) and \( W \), \( m \times n \) and \( n \times m \), respectively. It is shown in \([3]\) that there exists a unique matrix \( X \) \( m \times n \) satisfying:

(5.3) \( P(a)^{2n+1} x^{2n+1} = a^{2n+1}, \quad P(a)P(x) = P(x)P(a), \quad P(x)a = x \)
for some positive integer $k$. This matrix $X$ is called the \textit{W-weighted Drazin inverse} of $B$. It is easy to see that conditions (5.4) are equivalent to conditions (5.2) in the associative algebra of all complex rectangular matrices $m$-by-$n$, relative to the $W$-homotope product $M \circ N = MWN$, for $M, N$ $m$-by-$n$ matrices. Thus the $W$-weighted Drazin inverse of $B$ coincides with the usual Drazin inverse of $B$ relative to the (associative) $W$-homotope product.

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