Characterization of lifetime distributions in the $L_s$-sense by a generalized spread

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Abstract

We characterize partial orderings between lifetime distributions by $L_s$-tailweight. We obtain some ageing properties by comparing distributions with the negative exponential distribution with unit mean.

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1. Introduction

In the literature several concepts of partial orderings between lifetime distributions have been studied. These orderings are very important in reliability and economics applications. Barlow and Proschan (1975) and Stoyan (1983) have studied reliability or stochastic processes using these partial orderings.


Fernandez-Ponce (1993) characterized partial orderings in $L_1$-sense and different ageing properties by the spread functions. The spread function of a random variable and its properties are shown in Munoz (1990). In Section 2 we introduce the $u$-quantile spread of the tails in $L_s$-sense. In Section 3 we characterize three partial orderings in the $L_s$-sense among lifetime distributions by a generalized spread.

In Section 4 we obtain different ageing properties, characterizing them by generalized spread. These ageing properties are consequences of the previous partial orderings. Also, they can be obtained by comparing any...
distribution with the exponential distribution with unit mean. In this way, the different \( s \)-classes are obtained. Averous and Meste (1989) and Fagiuoli and Pellerey (1993) showed the connections among different classes.

2. Previous notions and definitions

Let \( \mathcal{F} \) be the family of absolutely continuous distributions \( F(x) \), such that \( F(0^+) = 0 \), with density function \( f(x) \) and with finite mean \( \mu_F \). The function \( \bar{F}(x) = 1 - F(x) \) is called the survival function, the function \( r_F(x) = f(x)/\bar{F}(x) \) is called its failure rate and \( Q_X(.) \) the left-continuous inverse (the quantile function). In the sequel also \( G \) belongs to \( \mathcal{F} \).

Let us give now some notations for the classic orderings (see Fagiuoli and Pellerey, 1993). We denote

\[
\bar{T}_{F_0}(x) = f(x) \quad \text{and} \quad \bar{T}_{F_s}(x) = \frac{\int_{x}^{\infty} \bar{T}_{F_{s-1}}(u) \, du}{\mu_{s-1}}, \quad \text{for } s \geq 1
\]

where \( \mu_s = \int_{0}^{\infty} \bar{T}_{F_s}(u) \, du \).

Also, we define

\[
r_{F_0}(x) = \frac{\bar{T}_{F_{s-1}}(x)}{\int_{x}^{\infty} \bar{T}_{F_{s-1}}(u) \, du} = \frac{(-d/dx)\bar{T}_{F_s}(x)}{\bar{T}_{F_s}(x)}, \quad \text{for } s \geq 1
\]

and

\[
r_{F_s}(x) = \frac{f'(x)}{f(x)}
\]

when \( f'(x) \) exists. Also, \( r_{F_s}(x) = \int_{0}^{x} r_{F_0}(t) \, dt \).

Averous and Meste (1989) defined the \( L_s \)-weight of the left tail, \( W_{s}^F(\infty, x) \) (resp. the right tail, \( W_{s}^F(y, \infty) \)). Now, we define the \( u \)-quantile spread of the left tail \((0, u)\) (resp. the right tail \((u, 1)\)) in the \( L_s \)-sense.

**Definition 2.1.** Let \( X \) be an absolutely continuous random variable with distribution function (d.f.) \( F_X(x) \). Then, the \( u \)-quantile spread of the left tail \((0, u)\) (resp. the right tail \((u, 1)\)) in the \( L_s \)-sense is

\[
A_{s}^F(u) = \int_{(0,u)} |Q_X(v) - Q_X(u)|^{s-1} \, dv.
\]

(resp.)

\[
V_{s}^F(u) = \int_{(u,1)} |Q_X(v) - Q_X(u)|^{s-1} \, dv.
\]

Note that,

\[
W_{s}^F(\infty, Q_X(u)) = A_{s}^F(u) \quad \text{and} \quad W_{s}^F(Q_X(u), \infty) = V_{s}^F(u).
\]

Also, \( V_{s}^F(u) + A_{s}^F(u) = S_X(u) \), where \( S_X(u) \) is the spread function of \( X \) (see Munoz-Perez, 1990). Fernandez-Ponce (1993) defined the right-spread function of \( X \) as \( S_X^+(u) = V_{s}^F(u) \), i.e. the \( u \)-quantile spread of the right tail in \( L_2 \) sense, and characterized by \( S_X^+(u) \) some partial orderings between lifetime distributions.

**Proposition 2.1.** Let \( F \) be an absolutely continuous distribution with \( F(0^+) = 0 \). Then, the following equalities \( \forall u \in (0,1) \) are verified:

\[
R_{F_1}^{-1}(u) = T_{F_1}^{-1}(1 - e^{-u}),
\]
Proving, (a) We know \( R_F(x) = -\ln \Phi_F(x) \).

Making the transformation
\[
    u = R_F(x) \implies e^{-u} = \Phi_F(x)
\]

\[
    \implies T_F(x) = 1 - e^{-u}
\]

\[
    \implies x = T_F^{-1}(1 - e^{-u})
\]

\[
    \implies R_F^{-1}(u) = T_F^{-1}(1 - e^{-u})
\]

(b)
\[
\frac{dR_F^{-1}(u)}{du} = \left[ \frac{du}{dR_F^{-1}(u)} \right]^{-1} (m = R_F^{-1}(u))
\]

\[
= \left[ \frac{dR_F(m)}{dm} \right]_{m=R_F^{-1}(u)}^{-1} = \frac{1}{r_F(m)}_{m=R_F^{-1}(u)} = \frac{1}{r_F(R_F^{-1}(u))}
\]

(c)
\[
\Phi_F(Q_x(u)) = \int_{0}^{\infty} (t - Q_x(u))^{-1}dF(t)
\]

\[
= \frac{-(s - 1)\int_{0}^{\infty} (t - Q_x(u))^{-2}dF(t)}{V_s^F(0)}
\]

(d)
\[
r_F(Q_x(u)) = \frac{d\Phi_F(Q_x(u))}{dQ_x(u)} / \Phi_F(Q_x(u))
\]

\[
= \frac{-dV_s^F(u)}{dQ_x(u)} / V_s^F(Q_x(u)).
\]
Note that
\[
\nu_s^F(Q_x(u)) = \frac{-1}{s} \frac{dV_s^F(Q_x(u))}{dQ_x(u)}
\]
\[
= \frac{s dV_s^F(u)}{dV_{s+1}^F(u)}.
\]

3. Partial orderings in the \( L_s \)-sense

In the literature convex ordering, starshaped ordering and superadditive ordering between distributions have been studied (see Dharmadhikari and Joag-Dev, 1988). These partial orderings can be interpreted as ageing properties when they compare with exponential distributions. The convex ordering is then equivalent to the IFR ageing property, the starshaped ordering to the IFRA property and the superadditive ordering to the NBU property.

Now, we define these ageing properties in the \( L_s \)-sense (see Fagiuoli and Pellerey, 1993) and characterize them by the \( u \)-quantile spread of the right tail.

Let \( X, Y \) be absolutely continuous nonnegative random variables with distribution functions \( F \) and \( G \), respectively.

**Definition 3.1.** \( Y \) is said to be larger than \( X \) in \( s \)-IFR ordering \( X \preceq_{s-IFR} Y \) if \((T_{G_{s-1}} \circ T_{F_s})(x)\) is convex.

**Definition 3.2.** \( Y \) is said to be larger than \( X \), in \( s \)-IFRA ordering \( X \preceq_{s-IFRA} Y \), if
\[
\frac{(T_{G_{s-1}} \circ T_{F_s})(x)}{x}
\]
is nondecreasing.

**Definition 3.3.** \( X \) is said to be larger than \( Y \) in \( s \)-NBU ordering \( X \preceq_{s-NBU} Y \) if \((T_{G_{s-1}} \circ T_{F_s})(x)\) is superadditive.

It is easily shown that
\[
(T_{G_{s-1}} \circ T_{F_s})(x) = (R_{G_{s-1}} \circ R_{F_s})(x)
\]
and
\[
\preceq_{s-IFR} F \preceq_{s-IFRA} G \Rightarrow F \preceq_{s-NBU} G \Rightarrow F \preceq_{s-NBU} G
\]

**Theorem 3.1.** \( F \preceq_{s-IFR} G \) iff
\[
\frac{\nu_{s-1}^F(u)}{V_s^F(u)} \cdot \frac{V_s^G(u)}{\nu_{s-1}^G(u)}
\]
is nondecreasing in \( u \), for any \( s \geq 2 \).
Proof. \( F \preceq s^{-IFR} G \) iff \( R_{G_s}^{-1}R_{F_s}(x) \) is convex,

\[
\text{iff } \begin{cases}
(r_{F_s} \circ R_{F_s}^{-1})(u) \\
(r_{G_s} \circ R_{G_s}^{-1})(u)
\end{cases}
\text{ is nondecreasing in } u,
\]

\[
\text{iff } \frac{dV^G_{s+1}(u)}{dV^G_s(u)} \cdot \frac{dV^F_{s+1}(u)}{dV^F_s(u)} \text{ is nondecreasing in } u.
\]

Using \( dV^G_s = -sq_Y(u)V^G_s(u)du \), where \( q_Y(u) = dQ_Y(u)/du \), we get the result.

Note that when \( s = 1 \) we obtain the convex ordering. Note that \( dV^F_1(u) = -du \).

**Theorem 3.2.** \( F \preceq s^{-IFRA} G \) iff

\[
T_{F_s}^{-1}(u) \cdot \frac{(V^F_{s-1} \circ F \circ T_{F_s}^{-1})(u)}{V^F_s(0)} \geq T_{G_s}^{-1}(u) \cdot \frac{(V^G_{s-1} \circ G \circ T_{G_s}^{-1})(u)}{V^G_s(0)}, \quad s \geq 2.
\] (4)

Proof. \( F \preceq s^{-IFRA} G \) iff \( (T_{G_s}^{-1} \circ T_{F_s})(x)/x \) is nondecreasing,

\[
\text{iff } \frac{d(R_{G_s}^{-1} \circ R_{F_s})(x)}{dx} \geq \frac{(R_{G_s}^{-1} \circ R_{F_s})(x)}{x}.
\]

Note that \( R_G(x) = -\ln T_G(x) \) and using (a) in the following equivalence

\[
F \preceq s^{-IFRA} G \iff \frac{dR_{G_s}^{-1}(t)}{dR_{F_s}^{-1}(t)} \geq \frac{R_{G_s}^{-1}(t)}{R_{F_s}^{-1}(t)},
\]

we get the result.

When \( s = 1 \) we obtain the starshaped ordering and the above equivalence would be expressed by

\[
F \preceq IFRA G \iff \frac{1}{Q_Y(u)} \cdot \frac{dV^F_s(u)}{du} \leq \frac{1}{Q_Y(u)} \cdot \frac{dV^G_s(u)}{du}.
\]

This last equivalence was proved by Fernandez-Ponce (1993).

**Theorem 3.3.** Let \( F \) and \( G \) be two distribution functions with finite moments. Suppose

\[
T_{G_s}^{-1}(u) \geq T_{F_s}^{-1}(u) \quad \forall u \in (0,1).
\] (e)

\[
\frac{(V^F_s(0))^2}{V^F_{s-1}(0)(V^F_s \circ F)(x)} \leq \frac{(V^G_s(0))^2}{V^G_{s-1}(0)(V^G_s \circ G)(x)}. \tag{f}
\]

Then, \( \forall s \geq 2 \quad \left( F \preceq s-NBU G \right) \) iff

\[
\frac{(V^F_{s-1} \circ F)(u)}{(V^F_s \circ F)(u)} \cdot \frac{V^F_s(0)}{V^F_{s-1}(0)} \geq \frac{(V^G_{s-1} \circ G)(u)}{(V^G_s \circ G)(u)} \cdot \frac{V^G_s(0)}{V^G_{s-1}(0)}. \tag{5}
\]
Proof. Firstly, we assume \( s \)-NBU. Then,

\[
(R_{G}^{-1} \circ R_{F})(x + y) \geq (R_{G}^{-1} \circ R_{F})(x) + (R_{G}^{-1} \circ R_{F})(y) \quad \forall (x, y) \in \mathbb{R}^2.
\]

Therefore, it is also verified

\[
\frac{(R_{G}^{-1} \circ R_{F})(x + y) - (R_{G}^{-1} \circ R_{F})(x)}{y} \geq \frac{(R_{G}^{-1} \circ R_{F})(y) - (R_{G}^{-1} \circ R_{F})(0)}{y}.
\]

When \( y \downarrow 0 \) we get the inequality:

\[
\frac{d(R_{G}^{-1} \circ R_{F})(t)}{dt} \bigg|_{t=x} \cdot \frac{dR_{F}(t)}{dt} \bigg|_{t=x} \geq \frac{d(R_{G}^{-1} \circ R_{F})(t)}{dt} \bigg|_{t=0} \cdot \frac{dR_{F}(t)}{dt} \bigg|_{t=0}
\]

Now, using (b) and (d) and applying \( u = (R_{G}^{-1} \circ R_{F})(x) \), we get the result.

Now, we assume the inequality is verified. Then, the following inequality is also true \( \forall x > 0 \),

\[
\frac{1}{x} \cdot \frac{(V_{s-1}^{-1} \circ G)(x)}{(V_{s}^{-1} \circ G)(0)} \cdot \frac{(V_{s}^{-1} \circ G)(0)^2}{(V_{s-1}^{-1} \circ G)(0)(V_{s}^{-1} \circ G)(x)} \leq \frac{1}{x} \cdot \frac{(V_{s-1}^{-1} \circ F)(x)}{(V_{s}^{-1} \circ F)(0)} \cdot \frac{(V_{s}^{-1} \circ F)(0)^2}{(V_{s-1}^{-1} \circ F)(0)(V_{s}^{-1} \circ F)(x)}.
\]

If we put \( x = T_{G}^{-1}(u) \), then applying (5), \( x \geq T_{F}^{-1}(u) \) and based on the hypothesis of the theorem and on the fact that \((V_{s}^{-1} \circ F)(x)\) is nonincreasing in \( x \), we get \( F \leq G \), then \( F \leq \leq \leq \leq \leq G \).

When \( s = 1 \) we obtain the superadditive ordering. Let \( X \) and \( Y \) be nonnegative absolutely continuous random variable. Suppose \( Q_{X}(u) \leq Q_{Y}(u) \forall u \in (0, 1) \) and \( q_{X}(0) \geq q_{Y}(0) \). Then,

\[
X \leq Y \quad \text{iff} \quad \left[ (V_{2}^{F})'(0) \right] \left[ (V_{2}^{G})'(u) \right] \leq \left[ (V_{2}^{G})'(0) \right] \left[ (V_{2}^{F})'(u) \right].
\]

4. Applications in ageing properties

Let \( X \) be an absolutely continuous nonnegative random variable representing the lifetimes of a component or system of components or any other subject. We always assume that the d.f. of \( X \) has at least finite \( s \)-order moments with respect to the origin.

We define different ageing properties in the \( L_{s} \)-sense using the partial orderings in the above section. The definitions are based on the comparison between the negative exponential distribution with mean 1 and the corresponding distribution. We characterize these ageing properties by the \( u \)-quantile spread of the right tail.

Definition 4.1. Let \( f \) and \( g \) be two functions in an interval \( A \subset \mathbb{R} \). \( f \) is said to be more bulging than \( g \) in \( A \) \( f \gg g \) if the function \( \phi(x) = f(x)/g(x) \) is nonincreasing in \( x \) \( \forall x \in A \).

Note 4.1. Let \( X \) and \( Y \) be two random variables representing lifetimes with density function \( f \) and \( g \), respectively. It is easy to show that \( Y \gg X \) iff \( f \gg g \). Note that \( \gg \) is the likelihood ratio ordering (see Fagiuloli and Pellerey, 1993).
Definition 4.2. Let $X$ be an absolutely continuous random variable representing the lifetime: it is said to be IFR in the $L_s$-sense (s-IFR), if $r_{F}(x)$ is nondecreasing in $x$, $x \geq 0$.

Theorem 4.1. $X$ is s-IFR iff
\[ \frac{dV^F_s(u)}{du} \geq \frac{dV^F_s(t)}{du}, \quad s \geq 2. \] (6)

Proof. In Section 2, we have seen that $r_{F}(x) = \frac{(-d/dx)T_{F}(x)}{T_{F}(x)}$ and if we do the change $F_{F}(x) = u$, then we get the following equivalence

$X$ is s-IFR iff $r_{F}(u) = \frac{dV^F_s(u)}{dV^F_s(t)}$ is nondecreasing in $u$.

Then, by Definition 4.1, we get the result.

Corollary 4.2. $X$ is s-IFR $\iff X \leq \exp(\lambda)$ for a single value of $\lambda$ (e.g. $\lambda = 1$), $s \geq 2$.

Proof. By Definition 3.1, we get that

$X \leq \exp(\lambda)$ iff $(R_{G_{s}}^{-1} \circ R_{F})$ is convex.

We know that if $Y \sim \text{Exp}(1)$ then $R_{G_{s}}^{-1}(t) = t$. Therefore, $d^2R_{F}(x)/dx \geq 0$, but it is $R_{F}(x) = \int_{0}^{x} r_{F}(t) dt$. Then, $dr_{F}(x)/dx \geq 0$, which means $r_{F}(x)$ is nondecreasing in $x \geq 0$.

Note 4.2. This proof is not valid for $s = 1$. Fernandez-Ponce (1993) showed that $F$ is IFR if, and only if, the $u$-quantile spread of the right tail for $s = 2$ is a convex function.

Definition 4.3. $X$ is said to be an s-IFRA distribution if $r_{F}(x)/x$ is non-decreasing in $x \geq 0$.

Corollary 4.3. $X$ is s-IFRA $\iff X \leq \exp(\lambda)$ for a single value of $\lambda$ (e.g. $\lambda = 1$), $s \geq 2$.

Proof. It is trivial using the Definition 3.2.

Corollary 4.4. $X$ is s-IFRA $\iff$
\[ T_{F_{s}}^{-1}(u) \cdot \frac{(V_{s}^{-1} \circ F \circ T_{F_{s}}^{-1}(u))}{(V_{s}^{-1} \circ F)(0)} \geq - (1 - u) \ln(1 - u), \quad s > 1. \] (7)

Proof. Trivial. Using Theorem 3.2, if $Y \sim \exp(1)$ then $T_{G_{s}}(x) = e^{-x}$ and $V_{s}^{-1}(u) = (1 - u) \Gamma(s - 1)$, $s > 1$.

Definition 4.4. Let $f$ be a decreasing function in an interval $A \subset R$. It is said to be $PF_2$ on the right of $x_0 \in A$ if:

$f(x_0)f(t + x) \leq f(t)f(x_0 + x) \quad \forall x \geq 0, \forall t \geq x_0, \quad t \in A$.

Definition 4.5. Let $X$ be an absolutely continuous random variable representing the lifetime: it is said to be NBU in $L_s$-sense (s-NBU) if

$T_{F_{s}}(t + x) \cdot \tilde{T}_{F_{s}}(0) \leq T_{F_{s}}(t) \tilde{T}_{F_{s}}(x) \quad \forall x, t \geq 0$.
Theorem 4.5. X is s-NBU if, and only if,

\((V_s^F \circ F)\) is PF$_2$ on the right of zero.

Proof. Trivial, if we note that

\[
\bar{T}_F(x) = \frac{(V_s^F \circ F)(x)}{(V_s^F \circ F)(0)}.
\]

Example 4.1. Let X be a random variable with distribution function

\[F_X(x) = xI_{[0,1]}(x) + I_{[1, +\infty)}(x).\]

It is easy to show that

\[(V_s^F \circ F)(x) = \left(1 - \frac{x}{s}\right)I_{[0,1]}(x)\]

\((V_s^F \circ F)(x)\) is PF$_2$ on the right of zero: then X is s-NBU for \(s \geq 1\).

Definition 4.6. Let \(f\) and \(g\) be two continuous functions and decreasing in an interval \(A \subset \mathbb{R}\), with \(f(x) > 0\) and \(g(x) > 0\). \(f\) is said to be more bulging on the right of \(x_0 \in A\) than \(g\) if

\[
\frac{f(x)}{f(x_0)} \leq \frac{g(x)}{g(x_0)} \quad \forall x > x_0, \ x \in A.
\]

Definition 4.7. Let \(X\) be an absolutely continuous random variable representing the lifetime of a unit: it is said to be NBUFR in \(L_1\)-sense (s-NBUFR) if

\[
r_{F_t}(0) \leq r_{F_t}(x), \quad \forall x \geq 0.
\]

Theorem 4.6. X is s-NBUFR if, and only if,

\[
V_{s}^F(u) \overset{B_0((0,1))}{\geq} V_{s-1}^F(u), \quad s > 1.
\]

Proof. \(r_{F_t}(0) \leq r_{F_t}(x), \ \forall x \geq 0\), iff

\[
\frac{(d/dx)\bar{T}_F(x)|_{x=0}}{\bar{T}_F(0)} \geq \frac{(d/dx)\bar{T}_F(x)}{\bar{T}_F(0)}.
\]

Since

\[
\frac{d}{dQ_F(u)}\bar{T}_F(Q_F(u)) = \left(-s \frac{V_s^F(u)}{V_s^F(0)}\right),
\]

then, \(X\) is s-NBUFR iff

\[
\frac{V_{s-1}^F(0)}{V_s^F(0)} \leq \frac{V_{s-1}^F(u)}{V_s^F(u)}
\]

which is the same as \(V_{s}^F(u) \overset{B_0((0,1))}{\geq} V_{s-1}^F(u)\).
Note 4.3. This proof is not valid for \( s = 1 \) but Fernandez-Ponce (1993) showed that

\[
X \text{ is HNBUFR iff } \frac{dV_F^s(u)}{du} \geq \lim_{u \to 0} \frac{dV_F(1-u)}{du} \quad \forall u > 0.
\]

When \( s = 2 \), this theorem allows us to identify the 2-NBUFR distributions using the inequality \( V_F^2(u) \to V_F^1(u) \), that is the same as \( S_X(u) \leq \mu_X(1-u) \). This last inequality characterizes the NBUE distributions (see Fernandez-Ponce, 1993). Therefore, we can establish the equivalence between the 2-NBUFR class and the NBUE class. This equivalence is shown in Deshpande’s paper. Here, we justify it from another point of view.

Definition 4.8. Let \( X \) be an absolutely continuous random variable representing the lifetime of a unit: it is said to be NBAFR in \( L_s \)-sense (\( s \)-NBAFR) if

\[
r_F(0) \leq \int_0^x r_F(u) \frac{du}{x} \quad \forall x > 0.
\]

Theorem 4.7. \( X \) is \( s \)-NBAFR iff

\[
\lim_{u \to 0} \frac{(d/du)V_F^s(u)}{s(d/du)V_F^s(u)} \geq \frac{T_F^{-1}(u)}{-\ln(1-u)},
\]

(10)

Proof. We know

\[
r_F(u) = \frac{s(d/du)V_F^s(u)}{(d/du)V_F^s(u)},
\]

then

\[
\lim_{u \to 0} \frac{s(d/du)V_F^s(u)}{(d/du)V_F^s(u)} \leq \frac{1}{x} \int_0^x r_F(u) du.
\]

If we do the transformation \( x = T_F^{-1}(u) \), on the right-hand side of the last inequality, we get the result.

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