Short Note:
A nonstandard finite difference method for the one-dimensional advection equation in cylindrical-polar coordinates

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There exists a variety of finite difference methods for advection equations that are based on upwind or flux limiters, so that the numerical solution satisfies certain monotonicity and entropy conditions and is free of unphysical oscillations. Recently, Mickens introduced an exact difference expression for the spherical wave equation based on the analytical solution of the steady-state equation and an exact difference formula for the one-dimensional advection equation. His approach is ad hoc, and it can only be used for equations whose solutions can be easily obtained analytically.

In this short note an analytical approximation is used to obtain finite difference expressions for the spherical wave equation based on a piecewise linearization technique. Thus the method proposed here does not require that the analytical solution be known. Furthermore it can be used to obtain the finite difference expressions for other equations. In addition, if the solution is continuous, the piecewise linearization technique provided piecewise continuous solutions.

The analytical solution of equation (1) can be expressed as

$$u(r, t) = \frac{r-t}{r} f(r-t)$$

while finite difference expressions for equation (1) are the same as those for the standard one-dimensional advection equation for the dependent variable $ur$.

If the time derivative in equation (1) is discretized while the $r$ coordinate is kept continuous, equation (1) may be written as

$$\frac{u - u^0}{k} + \frac{\partial (ur)}{\partial r} = 0$$

where $k$ denotes the time step, $u = u(r, t^{n+1})$ and $u^0(r) = u(r, t^n)$.

The analytical solution of equation (4), which corresponds to a method of lines, is

$$u = \frac{C}{r} \exp\left(-\frac{r}{k}\right) + \frac{1}{k} \int \frac{s-r}{k} \exp\left(\frac{s-r}{k}\right) u^0(s) ds$$

where $C$ is an integration constant.

The integral in equation (5) may be difficult to carry out analytically; however, in the interval $[r_i, r_{i+1}]$, $u^0$ may be approximated by the first term of its Taylor's series expansion around $r_i$, i.e., $u^0 \approx u^0_i$, and equation (5) may be expressed, in that interval, as

$$ur \exp\left(\frac{r}{k}\right) = u_i \exp\left(\frac{r_i}{k}\right) + u^0_i \left(\exp\left(\frac{r}{k}\right) - r \exp\left(\frac{r_i}{k}\right) + k u^0_i \left(\exp\left(\frac{r}{k}\right) - \exp\left(\frac{r_i}{k}\right)\right)\right)$$

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where \( u_i = u(r_i, t^{n+1}) \) and \( r \in [r_i, r_{i+1}] \). Furthermore, with \( h_i = r_{i+1} - r_i \), equation (6) yields

\[
\begin{align*}
    u_{i+1}r_{i+1} - u_ir_i & = \left( - \frac{h_i}{k} \right) \\
    & = u_i^r \left( r_{i+1} - r_i \exp \left( - \frac{h_i}{k} \right) \right) \\
    & + ku_i^0 \left( \exp \left( - \frac{h_i}{k} \right) - 1 \right)
\end{align*}
\]

(7)

for \( r \in [r_i, r_{i+1}] \). Equation (7) is a nonstandard finite difference discretization of equation (1), which corresponds to an exponentially fitted scheme.

For \( h_i/k \ll 1 \), i.e., large Courant numbers, equation (7) can be written as

\[
u_{i+1}r_{i+1} - u_ir_i = O \left( \frac{h_i}{k} \right)
\]

(8)

Therefore for \( k = \infty \) equation (7) is the exact solution of the time-independent spherical wave equation.

For \( h_i/k \gg 1 \), i.e., small Courant numbers, equation (7) may be expressed as

\[
\frac{u_{i+1} - u_i^0}{h_i} + \frac{u_i^0}{r_{i+1}} = h.o.t.
\]

(9)

where \( h.o.t. \) denotes higher order terms. Equation (9) corresponds to the discretization of

\[
\frac{\partial u}{\partial t} + \frac{u}{r} = 0
\]

(10)

If \( h_i/k \gg 1 \) and \( k/r_{i+1} \ll 1 \), equation (7) reduces to

\[
u_{i+1} = u_i^0 + h.o.t.
\]

(11)

which is identical to the standard discretization of the one-dimensional advection equation with \( k = h_i \), i.e., if the Courant number is equal to one.

It must be pointed out that, if the solution is continuous, i.e., if no shock waves are formed, equation (6) provides approximate piecewise continuous solutions to equation (1). Furthermore, if the solution is sufficiently differentiable, \( u^o \) in equation (5) may be replaced by its Taylor's series expansion about \( r_i \) and the integrals may be performed analytically.

The results obtained up to here have been obtained by assuming that \( u^o = u_i^0 \), i.e., that a left expansion was used. Other expansions may also be employed. For example, if \( u^o = u_i^{n+1} \) is substituted into equation (5), i.e., if a right expansion is employed, the following expressions result, in \( [r_i, r_{i+1}] \),

\[
\begin{align*}
    u & = u_i^r \frac{r_{i+1} - r}{k} + u_i^{n+1} \left( 1 - \frac{r_i - r}{r} \exp \left( \frac{r_i - r}{k} \right) \right) \\
    & + ku_i^{n+1} \frac{\exp \left( \frac{r_{i+1} - r}{k} \right) - 1}{r} 
\end{align*}
\]

(12)

where \( u_i = u(r_i, t^{n+1}) \) and \( r \in [r_i, r_{i+1}] \). Furthermore, with \( h_i = r_{i+1} - r_i \), equation (12) yields

\[
\begin{align*}
    u_{i+1}r_{i+1} & = u_{i+1}^r \frac{r_{i+1}}{r} \exp \left( \frac{r_i}{k} \right) \\
    & + ku_{i+1}^r \left( \exp \left( \frac{r_i}{k} \right) - 1 \right)
\end{align*}
\]

(13)

which for \( h_i/k \ll 1 \) reduces to

\[
\frac{u_{i+1} - u_i^0}{h_i} + \frac{u_i^0}{r_{i+1}} = h.o.t.
\]

(14)

while for \( h_i/k \gg 1 \) it reduces to

\[
u_{i+1} = u_{i+1}^0 + h.o.t.
\]

(15)

which does not correspond to the physics of the problem.

It must be noted that the left expansion should be used in advection problems with positive speed, while the right one must be used when the advection speed is negative.

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References