INVISCID, SLENDER, ANNULAR LIQUID JETS

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Abstract—Regular perturbation expansions are used to analyse unsteady, inviscid, slender, incompressible (constant density), axisymmetric, annular liquid jets when the gases enclosed by and surrounding the jet are dynamically passive. Both inertia- and capillarity-dominated annular jets are considered. It is shown that, for inertia-dominated jets, closure of the leading order equations is achieved at second order in the perturbation parameter which is the slenderness ratio, whereas closure is achieved at first order for capillarity-dominated jets. The leading order equations are used to determine the fluid dynamics of steady annular jets.

1. INTRODUCTION
Annular liquid jets (Fig. 1) have found many applications in chemical engineering due to their relatively simple geometry and the volume that they may enclose. For example, Baird and Davidson (1962a) studied annular liquid jets to determine the dynamic surface tension of liquids and found that, when the Weber number exceeds one, long thin jets are obtained, while, when the Weber number is less than one, the jets take up a rounded form. At the transition point, i.e. when the Weber number is one, a discontinuity appears in the liquid sheet. The formulation employed by Baird and Davidson (1962a) in their analysis was essentially that of Bousinesq (1869a, b), i.e. they used the hydraulic approximation, assumed steady, thin, incompressible, inviscid fluids, and obtained equations along and normal to the mean radius of the annular liquid jet.

Baird and Davidson (1962b) proposed the use of annular liquid jets to measure the diffusivities of sparingly soluble gases in water by determining the absorption rate assuming equilibrium conditions at the gas–liquid interfaces and neglecting mass transfer resistances. If the diffusivities of sparingly soluble gases were accurately known, annular liquid jets could be used to determine the solubilities of these gases in liquids by simply determining or measuring the mass absorption rate by the liquid.

The enclosed volumes formed by annular liquid jets may also be used as chemical reactors for the reaction and control of toxic wastes, the direct reduction of zirconium from zirconium tetrachloride and sodium, scrubbing of radioactive and non-radioactive particulate and soluble materials, stack emission scrubbing, volatile metal-halide reduction processes, gas–liquid and liquid–liquid chemical reactions, etc. (Roidt and Shapiro, 1985).

Analytical and numerical studies of the potential application of annular liquid jets as chemical reactors for the combustion of toxic wastes were performed by Ramos (1988) who projected the equations developed by Boussinesq (1869a, b) along and normal to the jet's mean radius, onto a cylindrical polar coordinate system, included gravitational forces, surface tension on both interfaces, and pressure differences between the gas enclosed by and surrounding the annular liquid jet and obtained analytical solutions for long, thin, under-, non-, and over-pressurized annular liquid jets. These analytical solutions agree very well with those obtained numerically from the steady state governing equations for very long annular liquid jets and with experimental data (Ramos, 1990). Ramos and Pitchumani (1990) used the equations of Ramos (1988) to study isothermal gas absorption in annular liquid jets and obtained analytical solutions for both the gas concentration in the liquid and the mass absorption rate.

Previous formulations of steady state, incompressible, inviscid, thin, annular liquid jets (Ramos, 1988) cannot be easily generalized to unsteady and/or viscous jets. For this reason, the author developed an integral formulation based on the integration of the continuity and Navier–Stokes equations across the jet thickness which automatically calls for the interface boundary conditions (Ramos, 1992). For inviscid or ideal fluids, the velocity component normal to each interface must be continuous while the pressure difference across the interface must be balanced by surface tension. For thin and long jets, i.e. for small thickness-to-mean radius at the nozzle exit and mean radius-to-convergence length ratios, application of the mean value theorem of integration and use of Taylor's series expansions of the interface boundary conditions about the jet's mean radius, resulted in a system of four, one-dimensional, time-dependent, partial
differential equations for the jet’s radius, and mass per unit length, average radial and axial velocity components. These equations coincide with those derived by Boussinesq (1869a, b) for steady, long, annular liquid jets.

In this paper, perturbation methods are employed to determine the leading order equations which govern the fluid dynamics of unsteady, slender, inviscid, irrotational, incompressible, axisymmetric, annular liquid jets. The small perturbation parameter is the annular liquid jet’s slenderness ratio, i.e. the ratio of the jet’s mean radius at the nozzle exit to a characteristic axial distance which may be assumed to be the convergence length. The analysis presented in the paper is essentially a long-wave one which differs from several respects. First, the perturbation method is limited to small slenderness ratios, while the integral balance is not. Second, the perturbation method works directly with the differential form of the continuity and Euler equations, whereas the integral balance formulation deals with the integrated form of these equations. Third, the perturbation method yields the pressure as a function of the spatial coordinates and time, whereas a hydraulic, i.e. a uniform pressure, assumption was made by Ramos (1992). Fourth, the integral balance formulation provides partial differential equations for the jet’s mean radius, average axial and radial velocity components, and mass per unit length which come from the integration of the continuity and Euler equations and form the kinematic boundary conditions at the jet’s interfaces.

The fluids enclosed by and surrounding the annular liquid jet will be assumed to be dynamically passive, i.e. \( P^* \) and \( P^* \) will be assumed to be spatially uniform, since their density is, in general, much smaller than that of liquids.

In addition, one must provide initial conditions and boundary conditions at \( z^* = 0 \), i.e. at the nozzle exit. The boundary conditions must be obtained by matching the inviscid flow inside the nozzle with that of the free, annular jet. Since the flow inside the nozzle must satisfy the no-penetration condition at the solid walls, whereas the boundary conditions for the free, annular jet involve free surfaces, a transition from the no-penetration to the free-surface flow is expected. Such a transition is not considered in this paper where the interest lies in the region below the nozzle exit.

2. GOVERNING EQUATIONS

The fluid dynamics of unsteady, axisymmetric, incompressible (constant density), inviscid, irrotational, annular liquid jets is governed by the continuity and Euler equations and the irrotational condition, i.e.

\[
\frac{\partial u^*}{\partial z^*} + \frac{1}{r^*} \frac{\partial (r^* u^*)}{\partial r^*} = 0
\]

\[
\rho^* \left( \frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial z^*} + v^* \frac{\partial u^*}{\partial r^*} \right) = - \frac{\partial p^*}{\partial z^*} + \rho^* g^*
\]

\[
\rho^* \left( \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial z^*} + v^* \frac{\partial v^*}{\partial r^*} \right) = - \frac{\partial p^*}{\partial r^*} - \frac{\partial u^*}{\partial z^*}
\]

where the asterisk denotes dimensional quantities, \( r^* \) and \( z^* \) are the radial and axial coordinates, respectively. \( p^* \) is the pressure, \( \rho^* \) the density, \( g^* \) the gravitational acceleration, and \( u^* \) and \( v^* \) denote the liquid’s axial and radial velocity components, respectively.

Equations (1)–(4) are subject to the following kinematic and dynamic boundary conditions at the annular liquid jet’s interfaces

\[
v^*(R^*_i, z^*, t^*) = \frac{\partial R^*_i}{\partial z^*} + u^*(R^*_i, z^*, t^*) \frac{\partial R^*_i}{\partial z^*} \quad i = 1, 2
\]

\[
P^*_i - p^*(R^*_i, z^*, t^*) = \sigma^* \left\{ \frac{1}{\left[ R^*_i \left[ 1 + \left( \frac{\partial R^*_i}{\partial z^*} \right)^2 \right]^{1/2} \right]} - \frac{(\partial^2 R^*_i / \partial z^*^2)}{\left[ 1 + \left( \frac{\partial R^*_i}{\partial z^*} \right)^2 \right]^{3/2}} \right\}
\]

\[
p^*(R^*_i, z^*, t^*) - P^*_2 = \sigma^* \left\{ \frac{1}{\left[ R^*_2 \left[ 1 + \left( \frac{\partial R^*_2}{\partial z^*} \right)^2 \right]^{1/2} \right]} - \frac{(\partial^2 R^*_2 / \partial z^*^2)}{\left[ 1 + \left( \frac{\partial R^*_2}{\partial z^*} \right)^2 \right]^{3/2}} \right\}
\]

where \( R^*_i \) and \( R^*_2 \) are the radii of the jet’s inner and outer interfaces, respectively, \( \sigma^* \) is the liquid’s surface tension, and \( P^*_1 \) and \( P^*_2 \) denote the pressure of the gases enclosed by the annular liquid jet and surrounding, respectively.

The gases enclosed by and surrounding the annular liquid jet will be assumed to be dynamically passive, i.e. \( P^*_1 \) and \( P^*_2 \) will be assumed to be spatially uniform, since their density is, in general, much smaller than that of liquids.

In addition, one must provide initial conditions and boundary conditions at \( z^* = 0 \), i.e. at the nozzle exit. The boundary conditions must be obtained by matching the inviscid flow inside the nozzle with that of the free, annular jet. Since the flow inside the nozzle must satisfy the no-penetration condition at the solid walls, whereas the boundary conditions for the free, annular jet involve free surfaces, a transition from the no-penetration to the free-surface flow is expected. Such a transition is not considered in this paper where the interest lies in the region below the nozzle exit.
For long or slender, annular liquid jets, \( \varepsilon = R_i/L^* \ll 1 \), where \( R_i \) and \( L^* \) denote the jet’s mean radius at the nozzle exit and a characteristic axial distance, e.g. the convergence length, respectively. If the radial and axial coordinates are non-dimensionalized with respect to \( R^*_i \) and \( L^* \), respectively, the axial and radial velocity components with respect to \( U^*_i \) and \( E^*_r \), respectively, and the pressure with respect to \( p^*_i \), where \( E^*_r \) is a (constant) reference axial velocity component at the nozzle exit, eqs (1)–(7) become

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial z} &= - \frac{\partial p}{\partial x} + \frac{1}{F^*_r}, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial z} &= - \frac{\partial p}{\partial z}, \\
\frac{\partial u}{\partial z} &= 0, \\
v(R_i, z, t) &= u(R_i, z, t) = 0, \quad i = 1, 2
\end{align*}
\]

If there is no surface tension, i.e. the Weber number is infinite, and \( F_r = O(1) \), the liquid’s velocity components, the pressure and the jet’s mean radius at the inner and outer interfaces can be written in terms of \( \varepsilon \) as

\[
\begin{align*}
u &= u_0 + \varepsilon^2 u_2 + O(\varepsilon^4) \\
v &= v_0 + \varepsilon^2 v_2 + O(\varepsilon^4) \\
p &= p_0 + \varepsilon^2 p_2 + O(\varepsilon^4) \\
R_1 &= R_{10} + \varepsilon^2 R_{12} + O(\varepsilon^4) \\
R_2 &= R_{20} + \varepsilon^2 R_{22} + O(\varepsilon^4).
\end{align*}
\]

Expansion of the kinematic [eq. (12)] and dynamic [eqs (13) and (14)] boundary conditions at \( R_1 \) and \( R_2 \) in Taylor’s series around \( R_{10} \) and \( R_{20} \), respectively, and substitution of eqs (15)–(19) into eqs (8)–(14) result in a system of equations in powers of \( \varepsilon^2 \). Equating terms of \( O(\varepsilon^4) \) yields

\[
\begin{align*}
\frac{\partial u_0}{\partial z} + \frac{\partial (v_0 r)}{\partial r} &= 0, \\
\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial z} &= - \frac{\partial p_0}{\partial z} + \frac{1}{F_r}, \\
\frac{\partial p_0}{\partial r} &= 0, \\
\frac{\partial u_0}{\partial r} &= 0,
\end{align*}
\]

Equations (22) and (23) imply that \( p_0 = C(z, t) \) and \( u_0 = A(z, t) \), while eqs (22) and (25) require that, for mathematical compatibility, \( p_0 = C(z, t) = P_1 = P_2 \). The solution of eq. (20) is

\[
v_0 = - \frac{A'}{2} r + B
\]

where \( B \) is a function of \( z \) and \( t \) and the primes denote partial differentiation with respect to \( z \).

Substitution of eq. (26) into eq. (24) yields

\[
\begin{align*}
\frac{1}{2} \frac{\partial R_{10}}{\partial t} + \frac{1}{2} \frac{\partial (AR_{10})}{\partial z} &= \frac{1}{2} \frac{\partial R_{10}}{\partial t} + \frac{1}{2} \frac{\partial (AR_{10})}{\partial z}, \\
\frac{1}{2} \frac{\partial R_{20}}{\partial t} + \frac{1}{2} \frac{\partial (AR_{20})}{\partial z} &= \frac{1}{2} \frac{\partial R_{20}}{\partial t} + \frac{1}{2} \frac{\partial (AR_{20})}{\partial z}.
\end{align*}
\]

Subtraction of eq. (28) yields

\[
\frac{1}{2} \frac{\partial (R_{20} - R_{10})}{\partial t} + \frac{1}{2} \frac{\partial (AR_{20} - AR_{10})}{\partial z} = 0
\]

which is a continuity equation.
Equation (21) becomes
\[
\frac{\partial A}{\partial t} + A \frac{\partial A}{\partial z} = \frac{1}{F_i}. \tag{30}
\]
Under steady state conditions and for uniform flow at the nozzle exit, the solution of eq. (30) is the famous Torricelli's free-law formula, i.e.
\[
A = \left(1 + \frac{2z}{F_i}\right)^{1/2}. \tag{31}
\]
To \(O(e^2)\), the radial momentum equation and the dynamic boundary conditions become
\[
\frac{\partial p_2}{\partial r} = \left( \frac{\partial v_0}{\partial t} - u_0 \frac{\partial v_0}{\partial z} - v_0 \frac{\partial v_0}{\partial r} \right) p \tag{32}
\]
whose solution is
\[
p_2 = -\left( \frac{\partial B}{\partial t} + AB' \right) \ln r - \frac{B^2}{2r^2} - \frac{r^2}{8} \left( A'^2 - 2AA'' - 2\frac{\partial A'}{\partial t} \right) + D(z, t) \tag{34}
\]
where \(D\) can be calculated from the condition that [cf. eq. (33)] \(p_2(R_{i0}, z, t) = 0\) at \(i = 1, 2\). Furthermore, eqs (33) and (34) imply that
\[
\left( \frac{\partial B}{\partial t} + AB' \right) \ln \frac{R_{i0}}{R_{i0}} + \frac{B^2}{2} \left( \frac{1}{R_{i0}^2} - \frac{1}{R_{i0}^2} \right) + \frac{R_{i0}^2 - R_{i0}^2}{8} = 0 \tag{35}
\]
which together with eqs (28) and (30) provides a system of partial differential equations for \(A, B, R_{i0}\) and \(R_{i0}\). Once these values are determined, \(D\) may be calculated from eqs (33) and (34).

It must be pointed out that, since eq. (28) involves \(\partial R_{i0}/\partial z\) or \(\partial R_{i0}/\partial z\) while eq. (35) involves \(\partial B/\partial z\), second-order spatial derivatives of \(R_{i0}\) or \(R_{i0}\) with respect to \(z\) appear in eq. (35). These derivatives demand that \(R_{i0}(0, t), R_{i0}(0, t),\) and \(\partial R_{i0}(0, t)/\partial z\) or \(\partial R_{i0}(0, t)/\partial z\) be specified since eq. (28) relates \(\partial R_{i0}/\partial z\) with \(\partial R_{i0}/\partial z\) as indicated in Section 4.1.

The equations governing the leading order fluid dynamics of inertia-dominated, annular liquid jets for \((m, n) = (1, \infty)\), \((0, 1)\) and \((1, 1)\) may be obtained in a similar manner to the one employed in previous

Fig. 2. Shape of downward annular liquid jets. \([We = \infty\) (solid lines), 100 (dashed lines), 10 (dashed-dotted lines). For each Weber number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.]
paragraphs, and may be written as

\[
\frac{1}{2} \frac{\partial R_{10}^2}{\partial t} + \frac{1}{2} \frac{\partial (AR_{10}^2)}{\partial z} = B \tag{36}
\]

\[
\frac{1}{2} \frac{\partial (R_{30}^2 - R_{10}^2)}{\partial t} + \frac{1}{2} \frac{\partial (A(R_{30}^2 - R_{10}^2))}{\partial z} = 0 \tag{37}
\]

\[
\frac{\partial A}{\partial t} + A \frac{\partial A}{\partial z} = \frac{1}{G} \tag{38}
\]

\[
\frac{\partial B}{\partial t} = -AB' + \left( B^2 \left( \frac{1}{R_{30}^2} - \frac{1}{R_{10}^2} \right) + \frac{R_{20}^2 - R_{10}^2}{8} \right) \left( A^2 - 2AA'' - 2 \frac{\partial A'}{\partial t} + \frac{1}{H} \left( \frac{1}{R_{10}} + \frac{1}{R_{20}} \right) \right) \ln \frac{R_{20}}{R_{10}} \tag{39}
\]

where \((G, H) = (F, W_e = \infty), (\infty, W_e = \infty), (F, W), \) and \((\infty, W)\) with \(F\) and \(W\) of \(O(1)\), correspond to \((m, n) = (0, \infty), (1, \infty), (0, 1)\) and \((1, 1)\), respectively. These equations have the same form as those of a hydraulic model of annular liquid jets (Ramos, 1992) and can be solved by means of the numerical technique developed by the author (Ramos, 1993). That numerical technique maps the curvilinear geometry of the annular jet into a unit square and uses upwind differences for the convection terms, and an implicit method in time, and may be used to determine both the steady and unsteady dynamics of annular liquid jets.

It is important to point out that the asymptotic analysis of inertia-dominated, annular liquid jets indicates that, to leading order, mathematical compatibility requires that the pressure of the gases surrounding the liquid be identical to that of the gases enclosed by the annular liquid jet, and identical to that of the liquid. Furthermore, eq. (35) indicates that, in addition to \(A(0, t), R_{30}(0, t),\) and \(R_{20}(0, t)\), an additional boundary condition must be provided. Such a boundary condition may be the specification of either \(\partial R_{30}(0, t)/\partial z\) or \(\partial R_{20}(0, t)/\partial z\) since these partial derivatives are related through eq. (28).

In order to handle a pressure difference between the gases enclosed by and surrounding the annular jet, surface tension effects must be much larger than those considered in the previous sections as indicated in the next section.

2.2. \(We = O(1)\) and \(Fr = O(1)\)

The capillarity-dominated flow regime is characterized by \(We = O(1)\) and \(Fr = Fe^{-2m},\) where \(F = O(1)\) and \(m \geq 0,\) and may be analysed in exactly the same

Fig. 3. Shape of downward annular liquid jets. \([We = \infty\) (solid lines), \(100\) (dashed lines), \(10\) (dashed-dotted lines). For each Weber number, the annular liquid jet’s inner and outer radii correspond to the bottom and top curves, respectively.\]
manner as the inertia-dominated flow regime. Here, only the case $We = O(1)$ and $Fr = O(1)$, i.e. $m = 0$, is considered. For these values of the Weber and Froude numbers, eqs (15)–(19) may be used, eqs (20)–(24) and (26)–(29) hold, and eq. (25) must be replaced by the following ones:

$$P_1 - p_0(R_{10}, z, t) = \frac{1}{We R_{10}},$$

$$P_2 - p_0(R_{20}, z, t) = -\frac{1}{We R_{20}} \quad (40)$$

which together with eq. (22) require that

$$P_1 - P_2 = \frac{1}{We} \left( \frac{1}{R_{10}} + \frac{1}{R_{20}} \right), \quad (41)$$

i.e. the difference between the pressure of the gases enclosed by and that of the gases surrounding the annular liquid jet is balanced by surface tension.

Since according to eq. (22), the pressure at leading order is not a function of the radial coordinate and since the gases enclosed by and surrounding the annular liquid jet were assumed to be dynamically passive, substitution of eq. (40) into eq. (21) yields

$$\frac{\partial A}{\partial t} + A \frac{\partial A}{\partial z} = \frac{1}{Fr} - \frac{R_{10}}{We R_{10}^2} = \frac{1}{Fr} + \frac{R_{20}}{We R_{20}^2} \quad (42)$$

which together with eqs (28) and (41) yields a system of partial differential equations for $A, B, R_{10}$ and $R_{20}$. Note that eq. (41) implies that $R_{10}$ and $R_{20}$ are algebraically related since, for dynamically passive gases, $P_1$ and $P_2$ are only functions of time. Note also that, when the Weber number is $O(1)$, the leading order equations are closed at leading order, while the results of the previous section indicate that, for large Weber numbers, closure of the leading order equations is achieved at second order.

If $m \geq 1$, the gravitational acceleration does not affect the equation for the leading order axial velocity component, i.e. the first term in the right-hand side of eq. (42) is not present if $m \geq 1$.

The results presented in this section correspond to $We = O(1)$ and were derived using the liquid's axial velocity at the nozzle exit as the velocity scale. When surface tension is larger than inertia, the non-dimensionalization of the flow variables presented in the previous sections is not adequate, and scalings based on capillarity should be employed as indicated in the next section.

![Fig. 4. Shape of downward annular liquid jets. $[We = \infty$ (solid lines), 100 (dashed lines), 10 (dashed-dotted lines). For each Weber number, the annular liquid jet's and outer radii correspond to the bottom and top curves, respectively.]

3. SLENDER, CAPILLARY, ANNULAR LIQUID JETS

Since a large surface tension implies that the Weber number is small and since the square root of the Weber number is the ratio of a characteristic axial velocity at the nozzle exit, \( u^e \), to the capillary velocity, \( u^c = (\sigma^e / \rho^e R^e)^{1/2} \), a small Weber number implies that the capillary velocity is larger than the liquid's axial velocity at the nozzle exit. Therefore, asymptotic analyses of large surface tension, i.e. capillary, annular jets may be conveniently performed by non-dimensionalizing the liquid velocity with respect to the capillary velocity as indicated in the next paragraphs.

If the axial and radial coordinates, jet's radii, time and pressure are non-dimensionalized with respect to \( L^e, R^e, R^e, L^e / u^e \) and \( \rho^e u^e^2 \), respectively, so that eqs (2), (6) and (7) become, respectively,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{\partial u}{\partial r} = - \frac{\partial p}{\partial z} + \frac{B_o}{\varepsilon} \tag{43}
\]

\[
P_1 - p(R_1, z, t) = \frac{1}{R_1^e [1 + \varepsilon^2 (\partial R_1 / \partial z)^2]^{1/2}}
\]

\[
\frac{\varepsilon^2 (\partial^2 R_1 / \partial z^2)}{[1 + \varepsilon^2 (\partial R_1 / \partial z)^2]^{3/2}} \tag{44}
\]

while eqs (1) and (3)–(5) become eqs (8) and (10)–(12), where \( B_o = \rho^e g^e R^e / \sigma^e \) is the Bond number.

If \( B_o = \tilde{B} \epsilon \), where \( \tilde{B} = O(1) \), substitution of eqs (15)–(19) into eqs (8) and (10)–(12) and expansion of the kinematic boundary conditions at the interface in Taylor's series yield eqs (27)–(29), while eqs (43)–(45) become to \( O(\epsilon^5) \)

\[
\frac{\partial A}{\partial t} + A \frac{\partial A}{\partial z} = B_o - C' \tag{46}
\]

\[
P_1 - p_0(R_{1,0}, z, t) = \frac{1}{R_{1,0}}, \quad p_0(R_{2,0}, z, t) - P_2 = \frac{1}{R_{2,0}}. \tag{47}
\]

Since \( p_0 \) is not a function of \( r \) [cf. eq. (22)], eq. (47) implies that, for mathematical compatibility,

\[
P_1 - P_2 = \frac{1}{R_{1,0}} + \frac{1}{R_{2,0}} \tag{48}
\]
which implies that the difference between the pressure of the gases enclosed by and that of the gases surrounding the annular liquid jet is balanced by surface tension. Furthermore, since $P_1$ and $P_2$ may be only functions of time because the gases enclosed by and surrounding the annular liquid jet were assumed to be dynamically passive, eq. (48) implies that

$$\frac{1}{R_{10}^2} \frac{\partial R_{10}}{\partial z} = -\frac{1}{R_{20}^2} \frac{\partial R_{20}}{\partial z} \quad (49)$$

and eqs (22), (47) and (48) yield

$$\frac{\partial A}{\partial t} + A \frac{\partial A}{\partial z} = B_\star - C' = B_\star - \frac{1}{R_{10}^2} \frac{\partial R_{10}}{\partial z}$$

$$\frac{\partial R_{10}}{\partial z} = B_\star + \frac{1}{R_{20}^2} \frac{\partial R_{20}}{\partial z}. \quad (50)$$

Equations (28), (48) and (50) represent a system of partial differential equations for $A$, $R_{10}$ and $R_{20}$.

If $B_\star = B e^{m}$, where $B = O(1)$ and $m$ is an odd natural number greater than or equal to three, the gravitational acceleration does not affect the axial momentum equation to leading order and the first term in the right-hand side of eq. (50) is not present in that equation.

### 4. STEADY, ANNULAR LIQUID JETS

The equations presented in Sections 2.1, 2.2 and 3 have analytical solutions for steady, annular liquid jets as indicated in the next sections.

#### 4.1. Inertia-dominated, annular liquid jets

For steady jets, the solutions of eqs (37) and (38) are, respectively,

$$\frac{A(R_{20}^2 - R_{10}^2)}{2} = \beta \quad (51)$$

$$A = \left(1 + \frac{2z}{G}\right)^{1/2}, \quad (52)$$

where $\beta$ is a (constant) non-dimensional volumetric flow rate, and eq. (53) is Torricelli's free-fall formula.

For steady flows, eqs (36) and (39) may be written as

$$\psi' = B \quad (53)$$

$$\psi'' = -\frac{\beta A'' + A^2(R_{10}^2 - R_{20}^2)}{2A \ln \frac{R_{20}}{R_{10}}} \frac{2}{H} \left(\frac{1}{R_{10}} + \frac{1}{R_{20}}\right) \quad (54)$$

Fig. 6. Shape of downward annular liquid jets. [$Fr = 2$ (solid lines), 10 (dashed lines), 1 (dashed-dotted lines). For each Froude number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.]
Equations (53)–(55) indicate that the values of \( R_{2o}, R_{1o} \) and \( R'_{2o} \) or \( R'_{1o} \) must be specified at the nozzle exit, i.e. at \( z = 0 \). Alternatively, one may specify \( R_{o}, R'_{o} \) and \( b_{o} = R_{2o} - R_{1o} \) at the nozzle exit, where

\[
R_{o} = \frac{R_{1o} + R_{2o}}{2} \quad (56)
\]
is the annular liquid jet's mean radius at the nozzle exit. If this is done

\[
R_{1o} = R_{o} - \frac{b_{o}}{2}, \quad R_{2o} = R_{o} + \frac{b_{o}}{2} \quad (57)
\]

\[
R'_{1o} = \frac{R_{2o}R_{o}}{R_{o}} + \frac{A'(R_{2o} - R_{1o})}{2A} \quad (58)
\]

### 4.2. Capillarity-dominated, annular liquid jets

For conciseness, only the steady state solutions of the equations presented in Section 3 are presented here. The steady state solutions of eqs (28) and (50) are

\[
A = \left( 1 + \frac{2z}{G} + \frac{1}{R_{1o}} - \frac{1}{R_{1o}(0)} \right)^{1/2} \quad (59)
\]

where \( G = 1/B \) or \( \infty \) for \( B_{o} = e^{n} \bar{B} \) and \( B_{o} = e^{n} \bar{B} \) with \( m \geq 3 \), respectively. Equations (48), (51) and (59) may be used to obtain the following algebraic equation:

\[
P_{1} - P_{2} = \frac{1}{R_{1o}} \left( \frac{R_{1o}(1 + \frac{2z}{G} + \frac{1}{R_{1o}} - \frac{1}{R_{1o}(0)} \right)^{1/4}}{1 + \left( B + R_{1o}^{2}(1 + \frac{2z}{G} + \frac{1}{R_{1o}} - \frac{1}{R_{1o}(0)} \right)^{1/2}} \right)^{1/2} \quad (60)
\]

### 5. PRESENTATION OF RESULTS

The leading order equations derived in Section 2.1 have been used to analyse the fluid dynamics of steady, long or slender, annular liquid jets by means of an adaptive finite-difference method similar to the one developed by Ramos (1993). This method maps the curvilinear geometry of the jet into a unit square, and

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Fig. 7. Shape of upward annular liquid jets. \( W_{e} = \infty \) (solid lines), 100 (dashed lines), 10 (dashed-dotted lines). For each Weber number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.
the equations in the transformed domain are solved by means of finite-difference methods. The steady state results obtained with the adaptive technique were in remarkable good agreement with the solution of the equations presented in Section 4.1 which were solved by means of a fourth-order accurate Runge-Kutta method. Some sample results illustrating the behavior of steady, upward and downward, annular liquid jets are presented in the next two sections.

5.1. Steady, downward, annular liquid jets

Figures 2–4 show the annular jet's inner and outer radii as functions of the vertical coordinate z for $Fr = 2, R_{10} = 0.95, R_{20} = 1.05$ and $dR_{0}(0)/dz = -0.25, 0$ and 0.25, respectively, and several Weber numbers. In these figures, the values of the Weber number have been represented by means of solid, dashed and dashed-dotted lines, and the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively, for each Weber number. Figure 2 corresponds to inwardly directed flows at the nozzle exit and indicates that the larger the Weber number, the larger the convergence length. Figure 2 also shows the thickening of the annular liquid jet at the convergence point, i.e. at the point where $R_{10} = 0$.

Figure 3 corresponds to axially directed flows at the nozzle exit and indicates that the jet converges very slowly onto the symmetry axis for $We = \infty$. This result is to be expected since, for an axially directed jet at the nozzle exit without surface tension, the jet flows axially and accelerates according to Torricelli's free-fall law, while its thickness decreases. Figure 3 also shows that the effects of surface tension are significant for $We = 100$.

Figure 4 corresponds to outwardly directed jets at the nozzle exit and illustrates the importance of both the mean radius slope at the nozzle exit and the Weber number on both the convergence length and the shape of annular liquid jets. This figure shows that the annular liquid jet merges onto the symmetry axis to become a "solid", axisymmetric one for $We < 100$; no convergence, however, is observed for $We = \infty$ although the jet's concavity is towards the symmetry axis on account of the downward gravitational pull.

The results presented in Figs 2–4 indicate that the convergence length of downward, annular liquid jets increases as the Weber number and the slope of the jet's mean radius at the nozzle exit are increased. The increase in convergence length as the mean radius's slope at the nozzle exit is increased, is, however, small for $We = 10$. Figures 2–4 also indicate that the validity

![Fig. 8. Shape of upward annular liquid jets. ($We = \infty$ (solid lines), 100 (dashed lines), 10 (dashed-dotted lines). For each Weber number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.)](image-url)
of the slender, annular liquid jet approximation is questionable for \( We = 10 \) which corresponds to a convergence length approximately equal to 1. Figures 2–4 also show that the convergence length increases as the nozzle angle is increased from negative to positive values.

The effects of the gravitational acceleration on the shape of steady, downward, annular liquid jets are illustrated in Fig. 5 which shows the annular jet's inner and outer radii as functions of the vertical coordinate \( z \) for \( We = 100, R_{10} = 0.95, R_{20} = 1.05, dR_{0}(0)/dz = 0 \) and \( Fr = 1, 2 \) and 10, and indicates that the convergence length and the jet contraction at the nozzle exit increase as the gravitational acceleration is increased, i.e. as the Froude number is decreased. Note that no jet contraction at the nozzle exit is observed for \( Fr = 10 \). Figure 5 also indicates that the convergence length is a weakly increasing function of the inverse of the Froude number for \( We = 100 \).

Figure 6 presents the annular jet's inner and outer radii as functions of the vertical coordinate \( z \) for \( We = 100, R_{10} = 0.9, R_{20} = 1.1, dR_{0}(0)/dz = 0 \) and \( Fr = 1, 2 \) and 10. This figure indicates that the jet's contraction at the nozzle exit is important for \( Fr = 1 \) and 2. Comparison of Figs 5 and 6 illustrates that the jet's contraction at the nozzle exit and the convergence length increase as the annular liquid jet's thickness at the nozzle exit is increased. The convergence length also increases as the annular liquid jet's thickness at the nozzle exit is increased for outwardly directed flows; for inwardly directed flows at the nozzle exit, however, the convergence length decreases as the jet's thickness at the nozzle exit is increased.

The results presented in Figs 2–6 clearly indicate that the convergence length is much larger than the jet's mean radius at the nozzle exit for \( We > 10 \); therefore, the slenderness approximation employed in this paper is justified for Figs 2–6 for \( We > 10 \). On the other hand, long jets may be prone to instabilities which have not been considered in this paper.

5.2. Steady, upward, annular liquid jets

For upward, annular liquid jets, the gravitational acceleration has the opposite direction to that of the axial velocity component of the liquid at the nozzle exit. Upward, annular liquid jets may be analysed by means of the equations presented in this paper by using a negative gravitational acceleration, i.e. a negative Froude number.

Figures 7–9 show the annular jet's inner and outer radii as functions of the vertical coordinate \( z \) for

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Fig. 9. Shape of upward annular liquid jets. \([We = \infty \) (solid lines), 100 (dashed lines), 10 (dashed-dotted lines). For each Weber number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.]
$Fr = -2$, $R_{10} = 0.95$, $R_{20} = 1.05$ and $dR_0(0)/dz = -0.25$, 0 and 0.25, respectively, and several Weber numbers. Figure 7 corresponds to an inwardly directed flow at the nozzle exit and indicates that the convergence length increases as the Weber number is increased and is smaller than that of downward annular liquid jets with $Fr = 2$ (cf. Fig. 2). Figure 7 also shows the thickening of the annular liquid jet downstream of the nozzle exit. This thickening is due to the fact that, according to eq. (52), the leading order axial velocity component of the annular liquid jet is zero at $z = -Fr/2$ with $Fr = O(1)$. At this axial location, the jet's thickness becomes infinite. The results presented in Figure 7 show that, for inwardly directed flows at the nozzle exit, the annular jet merges onto the symmetry axis and the liquid's axial velocity component does not reach a zero value for $We > 100$.

Figure 8 corresponds to vertically upward jets while Fig. 9 corresponds to radially outward jets; both figures exhibit similar trends to those of Fig. 7. In particular, the axial velocity component reaches a zero value near $z = -Fr/2$ with $Fr = O(1)$, for $We \geq 100$. Figures 8 and 9 also show that the effects of surface tension are not as important as for downward, annular liquid jets (compare with Figs 3 and 4).

The effects of the Froude number on upward, steady, annular liquid jets are illustrated in Fig. 10 which shows the annular jet's inner and outer radii as functions of the vertical coordinate $z$ for $We = 100$, $R_{10} = 0.95$, $R_{20} = 1.05$, $dR_0(0)/dz = 0$ and $Fr = -1$, $-2$ and $-10$, and indicates that the convergence length increases as the absolute value of the Froude number is increased. This result is expected, for, according to eq. (52), the axial location at which the axial velocity component is zero, increases as the Froude number is increased for upward, annular liquid jets. Figure 10 also shows that, except for $Fr = -1$ and $-2$, the axial velocity component does not reach a zero value at $z = -Fr/2$. In fact, the annular jet merges on the symmetry axis at an axial distance approximately equal to 2.3 for $Fr = -10$.

Figure 11 shows the annular jet's inner and outer radii as functions of the vertical coordinate $z$ for $We = 100$, $R_{10} = 0.9$, $R_{20} = 1.1$, $dR_0(0)/dz = 0$ and $Fr = -1, 2$ and $-10$, and indicates that the convergence length decreases as the annular liquid jet's thickness at the nozzle exit is increased for $-Fr > 2$ (compare with Fig. 10). This result also holds for outwardly directed flows at the nozzle exit. For inward flows at the nozzle exit, the convergence length, however,
Inviscid, slender, annular liquid jets

1.6

1.4

1.2

1.0

0.8

0.6

0.4

0.2

0.0

0.5

1.0

1.5

2.0

2.5

3.0

Fig. 11. Shape of upward annular liquid jets. \( \text{Fr} = -2 \) (solid lines), -10 (dashed lines), -1 (dashed-dotted lines). For each Froude number, the annular liquid jet's inner and outer radii correspond to the bottom and top curves, respectively.

decreases as the jet’s thickness at the nozzle exit is increased.

The results presented in Figs 2–6 and 7–11 show clearly that, for the same boundary conditions at the nozzle exit and Froude number, the effects of the Weber number are much more important for downward, annular jets than for upward ones. This seems to be a consequence of the fact that the gravitational pull decelerates the axial motion of upward jets, whereas it accelerates it in downward jets. Figures 2–6 also indicate that the thickness of annular liquid jets first decreases and then increases as the convergence point is approached, while that of upward jets always increases downstream from the nozzle exit.

6. CONCLUSIONS

Perturbation methods are employed to analyse the fluid dynamics of inviscid, irrotational, incompressible, axisymmetric, slender, annular liquid jets subject to gravity and surface tension. The small parameter is the jet’s slenderness ratio which is the ratio of the jet’s mean radius at the nozzle exit to a characteristic axial dimension. For inertia- and capillarity-dominated jets, the velocity scale corresponds to a constant axial velocity at the nozzle exit and the capillary velocity, respectively. The inertia-dominated flow regime is controlled by the Froude and Weber numbers while the capillary one is controlled by the Bond number. The magnitude of these numbers defines the different flow conditions that may be encountered.

It is shown that, for the inertia-dominated flow regime, closure of the leading order equations in the perturbation parameter requires the use of the dynamic boundary conditions at second order in the perturbation parameter, whereas closure is obtained at leading order in the capillarity-dominated regime.

The dynamics of steady, long, annular liquid jets has been examined numerically by means of an adaptive finite-difference method that maps the curvilinear geometry of the annular liquid jet into a unit square. It is shown that the convergence length of both downward and upward, annular liquid jets increases as the Weber number and the slope of the jet’s mean radius at the nozzle exit are increased. The convergence length also increases as the Froude number is decreased and increased, respectively, for downward and upward jets, respectively. For both axially directed and outward flows at the nozzle exit, the convergence length increases as the jet’s thickness at the nozzle exit is increased; the opposite has been observed for inward flows at the nozzle exit.
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