RING STRUCTURE OF THE FLOER COHOMOLOGY OF $\Sigma \times \mathbb{S}^1$

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We give a presentation for the Floer cohomology ring $HF^*(\Sigma \times \mathbb{S}^1)$, where $\Sigma$ is a Riemann surface of genus $g \geq 1$, which coincides with the conjectural presentation for the quantum cohomology ring of the moduli space of odd degree rank two stable vector bundles on $\Sigma$ with fixed determinant. We study the spectrum of the action of $H^*(\Sigma)$ on $HF^*(\Sigma \times \mathbb{S}^1)$ and prove a physical assumption made in [1]. © 1999 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Let $\Sigma = \Sigma_g$ be a Riemann surface of genus $g \geq 1$ and let $\mathcal{N}_g$ denote the space of flat $SO(3)$-connections with nontrivial second Stiefel--Whitney class $w_2$ modulo the gauge transformations that can be lifted to $SU(2)$. This is a smooth symplectic manifold of dimension $6g - 6$ and it is the space considered in [6, 4, Section 3]. Alternatively, we can consider $\Sigma$ as a smooth complex curve of genus $g$ and $\mathcal{N}_g$ as the moduli space of odd degree rank two stable vector bundles on $\Sigma$ with fixed determinant, which is a smooth complex variety of complex dimension $3g - 3$. The symplectic deformation class of $\mathcal{N}_g$ only depends on the genus $g$ and not on the particular complex structure on $\Sigma$. We note that, following the conventions of [8], the moduli space of anti-self-dual instantons on $\Sigma \times \mathbb{C}P^1$ with charge $\kappa = 0$ is again isomorphic to $\mathcal{N}_g$.

We consider the following rings associated with the Riemann surface (we will always use $\mathbb{C}$-coefficients):

- $QH^*(\mathcal{N}_g)$ is the quantum cohomology of $\mathcal{N}_g$ (see [13]). This is well-defined since $\mathcal{N}_g$ is a positive symplectic manifold. As vector spaces, $QH^*(\mathcal{N}_g) = H^*(\mathcal{N}_g)$, but the multiplicative structure is different. The minimal Chern number of $\mathcal{N}_g$ is 2, so $QH^*(\mathcal{N}_g)$ is $\mathbb{Z}/4\mathbb{Z}$-graded (the grading comes from reducing mod 4 the $\mathbb{Z}$-grading of $H^*(\mathcal{N}_g)$). The ring structure of $QH^*(\mathcal{N}_g)$, called quantum multiplication, is a deformation of the usual cup product for $H^*(\mathcal{N}_g)$. It is associative and graded commutative. We remark that we do not introduce Novikov rings to define $QH^*(\mathcal{N}_g)$ as in [13, Section 8] (otherwise said, as $H^2(\mathcal{N}_g) = \mathbb{Z}$, we should introduce an extra variable $q$ of degree 4, then we equate $q = 1$).

- $HF^*_*(\mathcal{N}_g)$ is the symplectic Floer homology of $\mathcal{N}_g$ (with the symplectomorphism $\phi = id$). The symplectic manifold $\mathcal{N}_g$ is connected, simply connected and $\pi_2(\mathcal{N}_g) = \mathbb{Z}$ (see [4, introduction]), so the groups $HF^*_*(\mathcal{N}_g)$ are well-defined [5]. They are $\mathbb{Z}/4\mathbb{Z}$-graded. $HF^*_*(\mathcal{N}_g)$ is endowed with the pair of pants product [12], which is an associative and graded commutative ring structure. The symplectic Floer cohomology of $\mathcal{N}_g$, $HF^*_*(\mathcal{N}_g)$, is defined as the dual of the symplectic Floer homology. There is a Poincaré duality [12, remark 2.4] and a pairing $\langle \cdot, \cdot \rangle$.

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\( HF_\*(\Sigma \times S^1) \) is the instanton Floer homology of the three manifold \( Y = \Sigma \times S^1 \) for the \( SO(3) \)-bundle with second Stiefel-Whitney class \( w_2 = PD[S^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z}) \). This is defined in [6] and is \( \mathbb{Z}/4\mathbb{Z} \)-graded. We introduce a multiplication on \( HF_\*(Y) \) using a suitable four-dimensional cobordism [14, section 5]. Let \( X \) be the four manifold given as a pair of pants times \( \Sigma \), which is a cobordism between \( Y \) and \( Y \). This gives a map \( HF_\*(Y) \otimes HF_\*(Y) \rightarrow HF_\*(Y) \), which is an associative and graded commutative ring structure on \( HF_\*(Y) \). Again, the instanton Floer cohomology of \( Y \), \( HF^\*(Y) \), is the dual of \( HF_\*(Y) \). There is a natural isomorphism \( HF^\*(Y) \cong HF_\*(Y) \) \( \oplus \mathbb{C} \), where \( \oplus \mathbb{C} \) denotes \( \mathbb{C} \) with reversed orientation. As \( Y = \Sigma \times S^1 \) admits an orientation reversing self-diffeomorphism, we can identify \( HF^\*(Y) \cong HF^\*(Y) \) \( \oplus \mathbb{C} \), so \( HF^\*(Y) = HF^\*(\Sigma \times S^1) \).

**Theorem 1.** There are natural isomorphisms of vector spaces

\[
QH^\* (\Sigma_\phi) \cong HF^\*_{symp}(\Sigma_\phi) \cong HF^\* (\Sigma \times S^1).
\]

Moreover the first isomorphism respects the ring structures.

**Proof.** The second isomorphism is due to Dostoglou and Salamon [4, Theorem 10.1]. It is the particular case where one considers \( \phi = \text{id} : \Sigma \rightarrow \Sigma \), in which the mapping torus of \( \phi \) is \( \Sigma \times S^1 \) and the \( SO(3) \)-bundle has \( w_2 = PD[S^1] \). The first isomorphism is a standard result obtained by Floer [5]. In [12, Theorem 5.1] it is proved that the first isomorphism intertwines the products. \( \square \)

**Conjecture 2.** The second isomorphism in (1) is a ring isomorphism.

D. Salamon has informed the author that the adiabatic limit techniques of [4] can be extended to give a proof of Conjecture 2. He gave a lecture at Maryland in Spring 1994 on this issue. Many implications of this result to four-dimensional topology were given by Donaldson [3]. The author followed this program in [9] for small genus. Later he obtained very nice results on the behaviour of Donaldson invariants under the operation of connected sum along a Riemann surface [10, 11], exploiting only the isomorphism (1) as vector spaces.

Using physical methods, Vafa et al. [1] find a set of generators and relations for \( QH^\* (\Sigma_\phi) \). There are two main assumptions in their argument. The first one is Conjecture 2. The second one is that the spectrum of the action of \( H^\* (\Sigma) \) on \( HF^\* (\Sigma \times S^1) \) can be read off from the Donaldson invariants of \( \Sigma \times \mathbb{T}^2 \).

Later, Siebert and Tian [15] claimed to have found a mathematical proof for the presentation of \( QH^\* (\Sigma_\phi) \) given in [1] but they could not yet finish their program. In this paper, we prove that the set of generators and relations given in [1] is a presentation for \( HF^* (\Sigma \times S^1) \) (Theorem 16), following a method inspired in Siebert and Tian [15]. This together with completion of the work [15] will produce a proof of Conjecture 2 (although in a rather indirect way). We also prove the physical assumption on the spectrum of \( HF^\* (\Sigma \times S^1) \) in [1] (cf. Proposition 20).

We leave the implications of Theorem 16 to Donaldson invariants of four-manifolds (mostly in the case \( \mu = 1 \)) for future work.
2. RING STRUCTURE OF $H^*({\mathcal N}_g)$

Let us recall the known description of the homology of $\mathcal{N}_g$ \cite{7, 17}. Let $\mathcal{V} \to \Sigma \times \mathcal{N}_g$ be the universal bundle and consider the Künneth decomposition as in \cite{7}

$$c_2(\text{End}_0 \mathcal{V}) = 2a[\Sigma] + 4\psi - b$$

with $\psi = \sum c_{i_1 i_2}^g$, where $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a symplectic basis of $H_1(\Sigma; \mathbb{Z})$ with $\gamma_{i_1 i_2}^g = [\Sigma]$ for $1 \leq i \leq g$, and $\{\gamma_i^g\}$ is the dual basis of $H^1(\Sigma)$. In terms of the map $\mu: H_*(\Sigma) \to H^{*-*}(\mathcal{N}_g)$, given by $\mu(x) = -\frac{1}{2} p_1(g\omega)/x$ (here $g\omega \to \Sigma \times \mathcal{N}_g$ is the associated universal $SO(3)$-bundle, and $p_1(g\omega) \in H^2(\Sigma \times \mathcal{N}_g)$ its first Pontrjagin class), we have

$$\begin{align*}
a & = 2\mu(\Sigma) \in H^2 \\
c_i & = \mu(\gamma_i) \in H^3, \quad 1 \leq i \leq 2g \\
b & = -4\mu(x) \in H^4
\end{align*}$$

where $x \in H_0(\Sigma)$ is the class of the point, and $H^i = H^i(\mathcal{N}_g)$. These elements generate $H^*(\mathcal{N}_g)$ as a ring \cite{7, 18}. So there is a basis $\{f_s\}_{s \in \mathcal{S}}$ for $H^*(\mathcal{N}_g)$ with elements of the form

$$f_s = a^b c_i^m c_i^m \cdots c_i^m$$

for a finite set $\mathcal{S}$ of multi-indices of the form $s = (n, m; i_1, \ldots, i_r)$, $n, m \geq 0$, $r \geq 0$, $1 \leq i_1 < \cdots < i_r \leq 2g$. The mapping class group $\text{Diff}(\Sigma)$ acts on $H^*(\mathcal{N}_g)$, with the action factoring through the action of $\text{Sp}(2g, \mathbb{Z})$ on $\{c_i\}$. The invariant part, $H^*_I(\mathcal{N}_g)$, is generated by $a, b$ and $c = -2\sum_{i=0}^g c_i c_{i+g}$. Then

$$H^*_I(\mathcal{N}_g) = \mathbb{C}[a, b, c]/I_g,$$

(2)

where $I_g$ is the ideal of relations satisfied by $a, b$ and $c$. Here $\deg(a) = 2$, $\deg(b) = 4$, $\deg(c) = 6$. Actually, a basis for $H^*_I(\mathcal{N}_g)$ is given by the monomials $a^b c^m e^r$, $n + m + r < g$ (see \cite{17}). For $0 \leq k \leq g$, the primitive component of $\Lambda^k H^3$ is

$$\Lambda^k H^3 \ni \ker(c^{g-k+1}: \Lambda^k H^3 \to \Lambda^{2g-k+2} H^3).$$

Then the $\text{Sp}(2g, \mathbb{Z})$-decomposition of $H^*(\mathcal{N}_g)$ is \cite{7}

$$H^*(\mathcal{N}_g) = \bigoplus_{k=0}^g \Lambda^k H^3 \oplus \mathbb{C}[a, b, c]/I_{g-k}.$$

**Proposition 3** (Siebert and Tian \cite{17}). For $g = 1$, let $q_1^1 = a$, $q_2^1 = b$, $q_3^1 = c$. Define recursively, for $g \geq 1$,

$$\begin{align*}
q_{g+1}^1 & = aq_g^1 + g^2 q_g^2 \\
q_{g+1}^2 & = bq_g^1 + \frac{2g}{g+1} q_g^3 \\
q_{g+1}^3 & = cq_g^1
\end{align*}$$

Then $I_g = (q_1^g, q_2^g, q_3^g)$, for all $g \geq 1$.

**Proof.** This is the form of the relations given in \cite[Proposition 3.2]{17}.

\[\square\]

3. RING STRUCTURE OF $HF^*_g$

With the aid of the basis $\{f_s\}_{s \in \mathcal{S}}$ for $H^*(\mathcal{N}_g)$ we are going to construct a basis for $HF^*_g = HF^*(\Sigma \times \mathbb{S}^1)$ to understand its ring structure. We need to use the gluing
properties of the Floer homology of a three manifold. Put \( Y = \Sigma \times \mathbb{S}^1 \) and let \( w_2 = PD[\mathbb{S}^1] \in H^2(Y; \mathbb{Z}/2\mathbb{Z}) \). Let us recall \([11]\) that for a 4-manifold \( X \) we define \( \mathbb{A}(X) = \text{Sym}^*(H_0(X) \oplus H_2(X)) \otimes \wedge^* H_1(X) \). Let us state the result that we shall use.

**Proposition 4** \([2, 3, 9]\). For any smooth oriented four-manifold \( X^4 \) with boundary \( \partial X^4 = Y \), any \( w \in H^2(Y; \mathbb{Z}) \) with \( w|_Y = PD[\mathbb{S}^1] \) in \( H^2(Y; \mathbb{Z}/2\mathbb{Z}) \), and any \( z \in \mathbb{A}(X^4) \), we have defined a relative invariant \( \phi^w(X^4, z) \in HF^*(Y) \). These relative invariants enjoy the following gluing property, suppose \( X = X_1^4 \cup_Y X_2^4 \) is a closed four-manifold split into two open four manifolds \( X_i^4 \) with \( \partial X_1^4 = Y, \partial X_2^4 = -Y \), and \( w \in H^2(Y; \mathbb{Z}) \) satisfying \( w|_Y = PD[\mathbb{S}^1] \) in \( H^2(Y; \mathbb{Z}/2\mathbb{Z}) \). Put \( w_i = w|_{X_i} \). Then for \( z_i \in \mathbb{A}(X_i^4) \), \( i = 1, 2 \), we have

\[
D_X^{w, \Sigma}(z_1z_2) = \langle \phi^{w_1}(X_1^4, z_1), \phi^{w_2}(X_2^4, z_2) \rangle
\]

where \( D_X^{w, \Sigma} = D_X^w + D_X^{w+\Sigma} \) (\( D_X^w \) is the Donaldson invariant of \( X \) for \( w \), see also \([9, 10]\)). When \( b^+ = 1 \), the invariants are calculated for a long neck, i.e. we refer to the invariants defined by \([\Sigma]\).

Consider the manifold \( A = \Sigma \times D^2 \), with boundary \( Y = \Sigma \times \mathbb{S}^1 \), and let \( \Delta = pt \times D^2 \subset A \) be the horizontal slice with \( \partial \Delta = \mathbb{S}^1 \). Put \( w = PD[\Delta] \in H^2(A; \mathbb{Z}) \). Clearly \( \mathbb{A}(A) = \mathbb{A}(\Sigma) = \text{Sym}^*(H_0(\Sigma) \oplus H_2(\Sigma)) \otimes \wedge^* H_1(\Sigma) \). For every \( s \in \mathcal{S}, f_s = \alpha b^n c_{i_1} \cdots c_{i_n} \), define

\[
z_s = \Sigma^n x^n \gamma_{i_1} \cdots \gamma_{i_n} \in \mathbb{A}(\Sigma)
\]

\[
e_s = \phi^w(A, z_s) \in HF^*(Y) = HF^*_g
\]

(see we identify Floer homology and Floer cohomology through Poincaré duality). Then \( \{e_s\}_{s \in \mathcal{S}} \) is a basis for \( HF^*_g \). This is a consequence of \([11, \text{Lemma 21}]\). The product \( HF^*_g \otimes HF^*_g \to HF^*_g \) is given by \( \phi^w(A, z_s) \phi^w(A, z_r) = \phi^w(A, z_sz_r) \). Then \( \phi^w(A, 1) \) defines the neutral element of the product. As a consequence, the following elements are generators of \( HF^*_g \),

\[
\begin{align*}
\alpha &= 2 \phi^w(A, \Sigma) \in HF^2_g \\
\psi_i &= \phi^w(A, \gamma_i) \in HF^3_g, \quad 0 \leq i \leq 2g.
\end{align*}
\]

\[
\beta = -4 \phi^w(A, x) \in HF^4_g
\]

Note that there is an obvious epimorphism of rings \( \mathbb{A}(\Sigma) \to HF^*_g \).

**Theorem 5.** Denote by \( * \) the product induced in \( H^*(\mathcal{N}_g) \) by the product in \( HF^*_g \) under the isomorphism \( H^*(\mathcal{N}_g) \cong HF^*_g \) given by \( f_s \mapsto e_s, s \in \mathcal{S} \). Then \( * \) is a deformation of the cup-product graded modulo 4, i.e. for \( f_1 \in H^i(\mathcal{N}_g), f_2 \in H^j(\mathcal{N}_g) \), it is \( f_1 * f_2 = \sum_{r \geq 0} \Phi_r(f_1, f_2) \), where \( \Phi_r \in H^* \mathcal{N}_g \) and \( \Phi_0 = f_1 \cup f_2 \).

**Proof.** First, for \( s, s' \in \mathcal{S} \),

\[
\langle e_s, e_{s'} \rangle = D^{w, \Sigma}_{\Sigma \times \mathbb{C}P^1}(z_sz_{s'}) = 0,
\]

unless \( \deg(f_s) + \deg(f_{s'}) = 6g - 6 + 4r, r \geq 0 \), as these are the only possible dimensions for the moduli spaces of anti-self-dual connections on \( \Sigma \times \mathbb{C}P^1 \). Moreover, when \( \deg(f_s) + \deg(f_{s'}) = 6g - 6 \), the moduli space is \( \mathcal{N}_g \), so \( \langle e_s, e_{s'} \rangle = -\langle f_s, f_{s'} \rangle \in \mathcal{N}_g \) (the minus sign is due to the different convention orientation for Donaldson invariants).
Now let $f_s, f'_s$ be basic elements of degrees $i$ and $j$ respectively. Put $f_s f'_s = \sum c_if_i$ and $f_s f'_s = \sum d_if_i$. This means that $e_s e'_s = \sum \alpha_e \epsilon_i$. Write $e_s e'_s = \sum g_m$, where $g_m = \sum_{\deg(f_s) = m} \epsilon_i$. The homogeneous parts. Put $g_m = \sum_{\deg(f_s) = m} \epsilon_i$. Let $M$ be the maximum $m$ such that $g_m \neq 0$. Then there is $f_s$ of degree $6g - 6 - M$ such that $\langle g_m, f_s \rangle \neq 0$. Since $\langle g_m, f_s \rangle = \langle g_M, e_s' \rangle = \langle e_s, e_s' \rangle = D_{\Sigma \times \mathbb{C}P^1}(z_3 z_5 z_7)$, it is $\deg(f_s) + \deg(f'_s) + \deg(f_s') \geq 6g - 6$, i.e. $M \leq i + j$. Now for $m = i + j$, any $f_s$ of degree $6g - 6 - m$, it is $\langle g_m, f_s \rangle = -D_{\Sigma \times \mathbb{C}P^1}(z_3 z_5 z_7) = \langle f_s f_s', f_s' \rangle$. So $g_{i+j} = f_s f'_s$.

Finally, $\langle e_s e'_s, e_s' \rangle = D_{\Sigma \times \mathbb{C}P^1}(z_3 z_5 z_7) = 0$, whenever $\deg(f_s) + \deg(f'_s) + \deg(f'_s) \neq 6g - 6 (mod 4)$, so $g_{i+j} = 0$ unless $M \equiv i + j (mod 4)$.

Remark 6. The isomorphism in Theorem 5 is only an isomorphism of $H^*(\mathcal{M}_g)$ and $HF^*_g$ as vector spaces (dependent on the chosen basis $\{f_s\}_{s \leq r}$). In general, this is not a ring isomorphism (see Example 22), so we do not expect the isomorphism in (1) to coincide with that of Theorem 5.

There is again an action of Diff($\Sigma$) on $HF^*_g$ factoring through an action of $Sp(2g, \mathbb{Z})$ on $\{\psi_i\}$. The invariant part $(HF^*_g)_I$ is generated by $\alpha, \beta$ and $\gamma = -2\sum_{-\infty}^{\infty} \phi^m(A, \gamma, \beta, \alpha)$. The epimorphism $\mathbb{C}[\alpha, \beta, \gamma] \to (HF^*_g)_I$, $\alpha \to \phi^m(A, \alpha)$, allows us to write

$$HF^*(\Sigma \times S^1)_I = \mathbb{C}[\alpha, \beta, \gamma]/J_g,$$

where $J_g$ is the ideal of relations of $\alpha, \beta$ and $\gamma$. Now $\deg(\alpha) = 2, \deg(\beta) = 4, \deg(\gamma) = 6$, but $J_g$ is not a homogeneous ideal.

Lemma 7. Suppose $gJ_g \subset J_{g+1}$, for all $g \geq 1$. Then we have the $Sp(2g, \mathbb{Z})$-decomposition

$$HF^*(\Sigma \times S^1) = \bigoplus_{g=0}^{\infty} \Lambda^g H^3 \otimes \mathbb{C}[\alpha, \beta, \gamma]/J_{g-k}.$$

Proof. The isomorphisms in Theorem 1 respect the $Sp(2g, \mathbb{Z})$-action and hence induce isomorphisms on the invariant parts. Then $\dim(HF^*_g)_I = \dim H^*(\mathcal{M}_g)$, for all $g \geq 1$. Now the lemma is a consequence of the argument in the proof of [7, Proposition 2.2] and the discussion preceding it.

4. A PRESENTATION FOR $(HF^*_g)_I$

Theorem 5 and the arguments in [17, Section 2] imply that we can deform the relations of $H^*(\mathcal{M}_g)$ to get a presentation for $(HF^*_g)_I$. More explicitly,

Lemma 8. It is $(HF^*_g)_I = \mathbb{C}[\alpha, \beta, \gamma]/(R^1, R^2, R^3)$, where $R^1 = q^1 + \text{lower order terms of degrees } \deg q^1 - 4r, r > 0$, as polynomials in $\mathbb{C}[\alpha, \beta, \gamma]$ ($q^1$ are defined in Proposition 3).

Proof. Suppose first that $g \geq 2$. Granted Theorem 5, [17, Theorem 2.2] implies that $J_g = (R^1, R^2, R^3)$, where $R^1 = q^1 + \text{lower order terms of degrees } \deg q^1 - 4r, r > 0$, as polynomials in $\mathbb{C}[\alpha, \beta, \gamma]$ (it can always be arranged so that these elements are in the basis, as $g \geq 2$, see [16, Proposition 4.2]). So $R^1$ is equal to $q^1 + \text{lower order terms.}$ The case $g = 1$ is computed directly in Lemma 11.
As pointed out in [15], the key result to find the generators of $J_g$ is the following inclusion of ideals.

**Lemma 9.** $J_{g+1} \subset J_g$, for all $g \geq 1$.

**Proof.** We shall prove this through an excision argument for Donaldson invariants. Let $\Sigma_g$ be a Riemann surface of genus $g$ and consider

$$\Sigma_{g+1} \subset A_g = \Sigma_g \times D^2 \subset S = \Sigma_g \times \mathbb{C}P^1$$

where $\Sigma_{g+1}$ is given as $\Sigma_g$ with a trivial handle of genus 1 added internally. This means that we take a point $p \in \Sigma_g$ and a small 4-ball $B$ in $S$ centered at $p$ such that $B \cap \Sigma_g$ is a 2-ball in $\Sigma_g$. Then we substitute $B \cap \Sigma_g$ with an embedded 2-torus in $B$ with a small ball removed, whose boundary coincides with $\partial(B \cap \Sigma_g) = \partial B$. This produces $\Sigma_{g+1}$. Then the map $H_*(\Sigma_{g+1}) \to H_*(\Sigma_g)$ induces $\mathbb{A}(\Sigma_{g+1}) \to \mathbb{A}(\Sigma_g)$ which sends $(z, \beta, \gamma) \mapsto (z, \beta, \gamma)$. Let $A_{g+1} = \Sigma_{g+1} \times D^2 \subset A_g$ be a (small) tubular neighbourhood of $\Sigma_{g+1}$ in $S$. Now we put $S^0$ for the complement of the interior of $A_{g+1}$ in $S$, so that $S = S^0 \cup \Sigma_{g+1} \times \mathbb{C}P^1$. Recall that we have also the decomposition $S = A_g \cup \Sigma_g \times \mathbb{C}P^1$.

Now let $z \in J_{g+1}$. Then $\phi^w(\Sigma_{g+1} \times D^2, z) = 0$. So for any $w \in \mathbb{A}(\Sigma_g)$, $s \in \mathcal{S}$,

$$D_S^{(w, \Sigma)}(z, s) = \langle \phi^w(\Sigma_{g+1} \times D^2, z), \phi^w(S^0, s) \rangle = 0.$$ 

Therefore $D_S^{(w, \Sigma)}(z, s) = \langle \phi^w(\Sigma_g \times D^2, z), \phi^w(\Sigma_g \times D^2, z) \rangle = 0$, for any $s \in \mathcal{S}$. So $\phi^w(\Sigma_g \times D^2, z) = 0$ and $z \in J_g$.

**Theorem 10.** There are numbers $c_{g+1}, d_{g+1} \in \mathbb{C}$ such that, for all $g \geq 1$,

$$\begin{align*}
R_{g+1}^1 &= \varepsilon R_g^1 + g^2 R_g^2 \\
R_{g+1}^2 &= (\beta + c_{g+1}) R_g^1 + \frac{2g}{g + 1} R_g^2 \\
R_{g+1}^3 &= \gamma R_g^1 + d_{g+1} R_g^2
\end{align*}$$

**Proof.** We follow almost literally the argument of Siebert and Tian [15, Proposition 3.2]. As $R_g^1 \in J_{g+1} \subset J_g$ is a relation on degree $2g + 2$, it is a linear combination of $\varepsilon R_g^1$ and $R_g^2$. Looking at the leading terms (Proposition 3), we have $R_{g+1}^1 = \varepsilon R_g^1 + g^2 R_g^2$. Analogously, $R_{g+1}^2$ is a combination of $\beta R_g^1$, $\varepsilon R_g^1$ and $R_g^1$. Only the term $R_g^2$ has degree less than $2g + 4$, so $R_{g+1}^2 = \beta R_g^1 + (\gamma g + 1) R_g^2 + c_{g+1} R_g^1$, for an unknown coefficient $c_{g+1}$. In the same fashion, $R_{g+1}^3$ is a combination of $\gamma R_g^1$ plus a linear combination of $R_g^2$ and $\varepsilon R_g^1$. Adding a suitable multiple of $R_g^1$ (which is always allowed without loss of generality), we have $R_{g+1}^3 = \gamma R_g^1 + d_{g+1} R_g^2$.

**Lemma 11.** The starting relations (for $g = 1$) are $R_1^1 = \varepsilon$, $R_1^2 = \beta - 8$ and $R_1^3 = \gamma$.

**Proof.** $HF^*_1$ is of dimension 1, i.e. $HF^*_1 = \mathbb{C}$ (see [3, 9]). Let $S$ be the K3 surface and fix an elliptic fibration for $S$, whose fibre is $\Sigma = \mathbb{T}^2$. The Donaldson invariants are, for $w \in H^2(S; \mathbb{Z})$ with $w \cdot \Sigma = 1 \pmod 2$ (see [9]),

$$D_S^{(w, \Sigma)}(e^D) = - e^{-Q(D)/2}.$$ 

Then $D_S^{(w, \Sigma)}(1) = -1$ and $D_S^{(w, \Sigma)}(\Sigma^d) = 0$, for $d > 0$. Also from [11, Remark 4], $D_S^{(w, \Sigma)}(\Sigma) = 2$. Let $S^0$ be the complement of an open tubular neighbourhood of $\Sigma$ in $S$. Then $\phi^w(S^0, 1)$
generates $HF^*_g$ and $\phi^*(S^*, \Sigma) = 0$, $\phi^*(S^*, x) = -2\phi^*(S^*, 1)$ and $\phi^*(S^*, \gamma_1\gamma_2) = 0$, i.e. $x = 0$, $\beta - 8 = 0$ and $\gamma = 0$ in $HF^*_g$ (recall (4)).

**Proposition 12.** For $g \geq 2$, there exists a non-zero vector $v \in HF^*_g$ such that

$$xv = \begin{cases} 4(g - 1)v, & g \text{ even} \\ 4(g - 1)\sqrt{-1}v, & g \text{ odd} \end{cases}$$

$$\beta v = (-1)^g - 18v$$

$$\gamma v = 0.$$  

*Proof.* We shall construct such a vector as the relative invariants of an open four-manifold $X^*$ with boundary $\partial X^* = Y = \Sigma \times S^1$, where the closed four-manifold $X = X^* \cup \gamma A$ is of simple type with $b^+ > 1$ and $b_1 = 0$. For concreteness, let $X$ be the manifold $C_g$ from [11, Definition 25]. We recall its construction. Let $S_g$ denote the elliptic surface of geometric genus $p_g = g - 1$ and with no multiple fibres. It contains a section $\sigma$ which is a rational curve of self-intersection $-g$. Let $F$ be the elliptic fibre. Then $\sigma + gF$ can be represented by an embedded Riemann surface $\Sigma$ of genus $g$ and self-intersection $g$. Blow-up $S_g$ at $g$ points in $\Sigma$ to get $B_g$ with an embedded Riemann surface $\Sigma_g$ of genus $g$ and self-intersection zero. Then put $X = C_g = B_g \# \Sigma_g$ (the double of $B$ along $\Sigma$). By [11, Proposition 27], $X$ is of simple type and $D^w_X(e^2) = D^w_X((1 + x/2)e^2) = -2^g - 5e^{Q(2)}/2e^{K_{X}} + (-1)^g2^g - 5e^{Q(2)}/2e^{-K_X}$, where $K \in H^2(X, \mathbb{Z})$ satisfies $K \Sigma_g = 2g - 2$ ($w \in H^2(C_g, \mathbb{Z})$ is a particular element, which we do not need to specify here). Let us suppose from now on that $g$ is even, the other case being similar. By [11, Proposition 3],

$$D^w_X((x^2) = -2^g - 5e^{Q(2)}/2e^{K_{X}} + (-1)^g2^g - 5e^{Q(2)}/2e^{-K_X}.$$ 

Consider $X^*$ the complement of a small open tubular neighbourhood of $\Sigma_g$ in $X$, so that $X = X^* \cup \gamma A$. Then we set $v = \phi^*(X^*, \Sigma + 2g - 2) \in HF^*(\Sigma \times S^1) = HF^*_g$. Let us prove that this is the required element. For any $z_s = \Sigma^a x^m \gamma_1 \cdots \gamma_k$, it is [11, Remark 4],

$$\langle v, e_s \rangle = D^w_X((\Sigma + 2g - 2)z_s) = \begin{cases} 0, & r > 0 \\ -2^g - 4(2g - 2)^{r - 1}2^m, & r = 0 \end{cases}$$

Then $\langle xv, e_s \rangle = \langle \phi^*(X^*, 2\Sigma (\Sigma + 2g - 2)), \phi^*(A, z_s) \rangle = D^w_X((\Sigma + 2g - 2)2 \Sigma z_s) = (4g - 4)\langle v, e_s \rangle$, for all $s \in S$. Then $xv = (4g - 4)v$. Analogously, $\gamma v = 0$ and $\beta v = -8v$. 

**Notation 13.** We set $R^1_0 = 1$, $R^2_0 = 0$ and $R^3_0 = 0$.

**Theorem 14.** For all $g \geq 1$, $c_g = (-1)^g 8$ and $d_g = 0$.

*Proof.* The result is true for $g = 1$ by Lemma 11 and Notation 13. Suppose it is true for $1 \leq r \leq g$, and let us prove it for $g + 1$. By Proposition 12, there exists $v \in HF^*_{g+1}$ with $\beta v = (-1)^g 8v$, $\gamma v = 0$ and $xv = 4q\sqrt{-1}v$ if $g$ is odd and $xv = 4q\sqrt{-1}v$ if $g$ is even.

In first place, $\gamma v = 0$ implies $R^3_2v = 0$, for $1 \leq r \leq g$. In second place, $\beta v = (-1)^g 8v$ implies

$$R^2_gv = (\beta + (-1)^g 8)R^1_{g-1}v = (-1)^g 16R^1_{g-1}v$$

$$R^2_{g-1}v = (\beta + (-1)^{g-1} 8)R^1_{g-2}v = 0$$
\[ R_{g-2}^2 = (-1)^g 16 R_{g-3}^4 \vphantom{R_{g-2}^2} \\
R_{g-3}^2 = 0, \]

In the third place, \( R_{g-1}^1 v = x R_{g-1}^1 v + (g-1)^2 R_{g-2}^2 v = x R_{g-1}^1 v, \) \( R_{g-2}^1 v = x R_{g-3}^2 v, \) \( \ldots \) Also
\[ R_{g-1}^1 v = x R_{g-2}^1 v + (g-2)^2 R_{g-2}^2 v = (x^2 + (g-2)^2(1-1)^2) R_{g-3}^2 v. \]

So finally,
\[ R_{g-1}^1 v = \begin{cases} (x^2 + (g-1)^2 16(g-2)^2 \ldots (x^2 + (g-1)^2 16 2^2) v, & g \text{ odd} \\ (x^2 + (g-1)^2 16(g-2)^2) \ldots (x^2 + (g-1)^2 16 2^2) v, & g \text{ even} \end{cases} \]

As a conclusion \( R_{g-1}^1 v = \lambda v, \) with \( \lambda \neq 0, \) and
\[ R_{g}^1 v = \lambda R_{g-1}^1 v \\
R_{g}^2 v = (-1)^g 16 R_{g-1}^2 v \\
R_{g}^3 v = 0. \]

As \( v \in HF^*_{g+1}, \) we have \( R_{g+1}^1 v = 0, R_{g+2}^2 v = 0 \) and \( R_{g+3}^3 v = 0. \) Evaluate the equations from Theorem 10 on \( v \) to get \( c_{g+1} = (-1)^g 18 + 8 \) and \( d_{g+1} = 0. \]

**Corollary 15.** We have \( \gamma J_g \subset J_{g+1} \subset J_g, \) for all \( g \geq 1. \)

**Proof.** The second inclusion is Lemma 9. For the first inclusion, note that \( \gamma R_{g}^1 = R_{g+1}^2 \in J_{g+1} \) by the third equation in Theorem 10. Then multiplying the first two equations in Theorem 10 we get that \( \gamma R_{g}^2, \gamma R_{g}^3 \in J_{g+1}. \)

Using this corollary in Lemma 7, we have finally proved that

**Theorem 16.** The Floer cohomology of \( \Sigma \times S^1, \) for \( \Sigma = \Sigma_g \) a Riemann surface of genus \( g, \) has a presentation

\[ HF^*(\Sigma \times S^1) = \bigoplus_{k=0}^g \Lambda_k^* H^3 \otimes C[z, b, \beta]/J_{g-k}. \]

where \( J_r = (R_r^1, R_r^2, R_r^3) \) and \( R_r^i \) are defined recursively by setting \( R_0^1 = 1, R_0^2 = 0, R_0^3 = 0 \) and putting for all \( r \geq 0 \)

\[
\begin{align*}
R_{r+1}^1 &= xR_r^1 + r^2 R_r^2 \\
R_{r+1}^2 &= (\beta + (-1)^r + 1) R_r^1 + \frac{2r}{r+1} R_r^2 \\
R_{r+1}^3 &= \gamma R_r^1.
\end{align*}
\]

**Remark 17.** The presentation obtained for \( HF^*_g \) is the conjectural presentation for \( QH^*(\Sigma_g) \) (see [15]).

**Corollary 18.** \( \ker(\gamma \cdot (HF^*_g)) \rightarrow (HF^*_g)) = J_{g-1}/J_g \subset C[z, b, \beta, \gamma]/J_{g} = (HF^*_g)_I. \)

**Proof.** By Corollary 15, \( \gamma \) factors as
\[ C[z, b, \gamma]/J_g \rightarrow C[z, b, \gamma]/J_{g-1} \rightarrow C[z, b, \gamma]/J_{g}. \]
The second map is a monomorphism since $x^a y^b z^c$, $a + b + c < g - 1$, form a basis for $\mathbb{C}[x, y, z]/J_{g-1}$, and their image under $\gamma$ are linearly independent in $\mathbb{C}[x, y, z]/J_g$. The corollary follows.

For any $F \in \mathbb{C}[x, y, z]$ define the expectation value by $\langle F \rangle_g = \langle F, 1 \rangle_{HF^*_g}$, where $1 \in HF^*_g$ is the unit element. Therefore $\langle F_1 F_2 \rangle_{HF^*_g} = \langle F_1, F_2 \rangle_{g'}$.

**COROLLARY 19.** For any $F \in \mathbb{C}[x, y, z]$, $\langle g F \rangle_g = 2g \langle F \rangle_{g-1}$.

**Proof.** By Corollary 15, the above formula holds for any $F \in J_{g-1}$, as both sides are zero. So it is enough to check it for a set of elements generating $HF^*_g$, i.e. for $F_{abc} = x^a y^b z^c$, $a + b + c < g - 1$. If $(a, b, c) \neq (0, 0, g - 2)$, it is $\langle F_{abc} \rangle_{g-1} = 0$ and $\langle g F_{abc} \rangle_g = 0$ by degree reasons. Now $\langle g^2 \rangle_{g-1} = -\langle x^2 \rangle_{g-1} = 0$ and $\langle g \rangle_{g-1} = 0$ by degree reasons. Therefore $\langle x^2 \rangle_{g-2} = 2g \langle x^2 \rangle_{g-3}$. Hence the corollary follows from $\langle x^2 \rangle_{g-2} = 2g \langle x^2 \rangle_{g-3}$. (See [18]).

### 5. LOCAL RING DECOMPOSITION OF $HF^*_g(\Sigma \times S^1)$

In [1] it is asserted that the only eigenvalues of the action of $\mu(x)$, $\mu(y)$ and $\mu(z)$ on $HF^*_g(\Sigma \times S^1)$ are the ones given by looking at the Donaldson invariants of the manifold $X = \Sigma \times \mathbb{T}^2$ i.e. if we denote by $W \subset HF^*_g$ the image $\phi(X, \partial(\Sigma))$, where $X = X^0 \cup \partial(X)$, then $x, y, z$ act on $W$ and their eigenvalues are all the eigenvalues of their action on $HF^*_g$. The following result is a proof of this physical assertion.

**PROPOSITION 20.** The eigenvalues of $(x, y, z)$ in $(HF^*_g)_f$ are $(0, 8, 0)$, $(\pm 4, -8, 0)$, $(\pm 8, 0, 8)$, $(\pm 8, \pm 1, \pm 8)$, $(\pm 4(g - 1), \pm 1, \pm 8)$, $(\pm 1\gamma, \pm 1, \pm 8)$, where $\gamma = 0, 1, 2, 3, 4, 5$.

**Proof.** Put $V = (HF^*_g)_f$. As $\gamma J_{g-1} \subset J_g$, one has $\gamma^g \in J_g$, i.e. $\gamma^g = 0$ in $V$, so the only eigenvalue of $\gamma$ is zero. To compute the eigenvalues of $x, y, z$ we can restrict to $V/\gamma V$ if $p$ is a polynomial with $p(x) = 0$ in $V/\gamma V$, then $p(x)$ is a multiple of $\gamma$ in $V$ and $p(x)^g = 0$ in $V$. Now the ideal of relations of $V$ can be written as $J_g = \langle \zeta_0, \zeta_1, \zeta_2 \rangle$, where $\zeta_0 = 0, 1, \zeta_{-1} = 0, \zeta_{-2} = 0$ and $\zeta_{r+1} = \zeta_r^2 + r^2(\beta + (-1)^g8)\zeta_{r-1} + 2r(r - 1)\gamma\zeta_{r-2}$, for all $r \geq 0$. This form of the relations follows from rewriting as

$$ R_1^g = \zeta_g $$

$$ R_2^g = \frac{1}{g^2}(\zeta_{g+1} - \zeta_g) $$

$$ R_3^g = \frac{1}{2g(g + 1)}(\zeta_{g+2} - \zeta_{g+1} - (g + 1)^2(\beta + (-1)^g8)\zeta_g). $$

Therefore

$$ V/\gamma V = \mathbb{C}[x, y, z]/(\zeta_{g+1}, \zeta_g) $$

where $\zeta_0 = 1, \zeta_{-1} = 0, \zeta_{r+1} = \zeta_r^2 + r^2(\beta + (-1)^g8)\zeta_{r-1}$, for $r \geq 0$. From $r^2(\beta + (-1)^g8)\zeta_{r+1} = \zeta_{r+1} - \zeta_r$ we infer that $(\beta + (-1)^g8)\zeta_{r-1} \in (\zeta_r, \zeta_{r+1})$. Continuing in this way,

$$(\beta + (-1)^g8)(\beta + (-1)^g8) \cdots (\beta - 8) \in (\zeta_{g}, \zeta_{g+1})$$
which implies that the only eigenvalues of $\alpha$ in $V/\gamma V$, and hence in $V$, are $\pm 8$. Let us study the eigenvalues of $\alpha$ for $\beta = 8$, $\gamma = 0$. Again we only need to study $V/\gamma (V, \beta - 8) = V = \mathbb{C}[x]/(\xi_0, \xi_{r+1})$, where now $\xi_0 = 1$, $\xi_{-1} = 0$, $\xi_{r+1} = \xi_r + r^2(8 + (-1)^8)\xi_{r-1}$, for $r \geq 0$. Then

$$
\begin{align*}
\xi_r &= (x^2 + (r-2)^2)16 \cdots (x^2 + 2^16)x^2, \quad r \text{ even} \\
\xi_r &= (x^2 + (r-1)^2)16 \cdots (x^2 + 2^16)x, \quad r \text{ odd}
\end{align*}
$$

from where the eigenvalues of $\alpha$ will be $0, \pm 8\sqrt{-1}, \pm 16\sqrt{-1}, \ldots, \pm 8[\sqrt{3-1}]\sqrt{-1}$. We leave the other case to the reader.

**Remark 21.** As mentioned in [1], by the very definition of $\gamma = -2\sum_{i=1}^{n}(A, \gamma, \gamma_1+a)$, it is $\gamma^{e+1} = 0$ in $HF^*_g$, so the only eigenvalue of $\gamma$ is zero.

Proposition 20 says that $(HF^*_g)_H$ can be decomposed as a sum of local artinian rings

$$(HF^*_g)_H = \bigoplus_{i=(g-1)}^{g-1} R_{g,i} \tag{6}$$

where $R_{g,i}$ is a local artinian ring with maximal ideal $m = (x - 4i, \beta - (-1)^8, \gamma)$ if $i$ is odd, $m = (x - 4i\sqrt{-1}, \beta - (-1)^8, \gamma)$ if $i$ is even. Also $HF^*_g$ is decomposed as

$$HF^*_g = \bigoplus_{k=0}^{g} \bigoplus_{i=(g-k-1)}^{g-k-1} \Lambda^k_0 H^3 \otimes R_{g-k,i} = \bigoplus_{k=0}^{g} \bigoplus_{i=(g-1)}^{g-1} \Lambda^k_0 H^3 \otimes R_{g-k,i} \tag{7}$$

We recall from Lemma 11 that $HF^*_1 = \mathbb{C}[x, \beta, \gamma]/(x, \beta - 8, \gamma)$. Let us see the next cases.

**Example 22.** For $g = 2$, $J_2 = (x^2 + \beta - 8, \alpha(\beta + 8), x\gamma)$. In $(HF^*_g)_H$, $\gamma = -x(\beta + 8)$ and $\gamma x = 0$. Hence $x^2(\beta + 8) = 0$. Now $x^2 = -x(\beta - 8)$ so $\beta(8 + \beta) = 0$ and $x^2(8 - 16) = 0$. Also $\gamma J_1 \subset J_2$ implies $\gamma x = \gamma^2 = \gamma(\beta - 8) = 0$. Finally $(\gamma + 16\gamma, x^2 - 16) = -16\gamma + 16(x^2 - 16) = -16(\gamma + x(\beta + 8)) = 0$. All together proves

$$(HF^*_g)_H = \mathbb{C}[x, \beta, \gamma]/(x^2, \beta - 8, \gamma + 16\gamma) \oplus \mathbb{C}[x, \beta, \gamma]/(x^2, \beta - 8, \gamma + 16\gamma) \oplus \mathbb{C}[x, \beta, \gamma]/(x^2, \beta - 8, \gamma + 16\gamma)$$

We want to remark that $HF^*_g \cong QH^*(A_2\gamma)$ (see [12, Example 5.3] for a presentation of the latter ring). The isomorphism sends $x \mapsto h_2, \beta \mapsto -4(h_4 - 1), \gamma \mapsto 4(h_6 - h_2)$, where $h_2, h_4, h_6$ are the generators of $QH^2, QH^4, QH^6$, respectively. This was conjectured in [9, Conjecture 1.22].

**Example 23.** For $g = 3$, $J_3 = (x(x^2 + \beta - 8) + 4(x\beta + 8x + \gamma), (\beta - 8) (x^2 + \beta + 8) + \frac{4}{3} x\gamma, \gamma(x^2 + \beta - 8))$. Put $V = (HF^*_g)_H$. Then

$$V/\gamma V = \mathbb{C}[x, \beta]/(x(x^2 + \beta - 8) + 4(x\beta + 8x), (\beta - 8)(x^2 + \beta - 8))$$

In $V/\gamma V$, the first relation yields $-5x(\beta - 8) = x^3 + 64x$ and the second $x(\beta - 8)(x^2 + \beta - 8) = 0$. This implies $x(x^2 - 16)(x^2 + 64) = 0$. Also $(\beta - 8)x(x^2 - 16) = 0$. Using $(\beta - 8)x^2 = -(\beta - 8)^2$, we get $x(\beta - 8)(\beta + 8) = 0$.

Therefore, in $V$, $x(x^2 - 16)(x^2 + 64)$ and $x(\beta - 8)(\beta + 8)$ are multiples of $\gamma$. As $\gamma J_2 \subset J_3$, we have $\gamma x(x^2 - 16) = 0$ and $\gamma x^2(\beta + 8) = 0$ by Example 22. So $x^2(x^2 - 16)(x^2 + 64) = 0$, 526 V. Muñoz
\[ x^3(x^2 - 16) (\beta - 8) (\beta + 8) = 0 \] and \[ x^3(\beta - 8) (\beta + 8)^2 = 0. \] It can be checked now that

\[
(HF^*_\beta)_I = \frac{\mathbb{C}[x, \beta, \gamma]}{(x - 8\sqrt{-1}, \beta - 8, \gamma)} \oplus \frac{\mathbb{C}[x, \beta, \gamma]}{((x - 4)^2, \gamma + 3(\beta + 8), \gamma - 12(x - 4))}
\]

\[
\oplus \frac{\mathbb{C}[x, \beta, \gamma]}{(x^3, x(\beta - 8), (\beta - 8)^2 - \frac{b^2}{4} x^2, \gamma + 16x)} \oplus \frac{\mathbb{C}[x, \beta, \gamma]}{((x + 4)^2, \gamma - 3(\beta + 8), \gamma - 12(x + 4))}
\]

\[
\oplus \frac{\mathbb{C}[x, \beta, \gamma]}{(x + 8\sqrt{-1}, \beta - 8, \gamma)}
\]

6. CONJECTURE

We state the following conjecture, that first occurred to Paul Seidel and the author in mid-1996.

**Conjecture 24.** The decomposition in equation (7) is

\[
HF^*_\beta \cong H^*(s^\beta \Sigma) \oplus H^*(s^1 \Sigma) \oplus \cdots \oplus H^*(s^g \Sigma)
\]

\[
\oplus H^*(s^{g-1} \Sigma) \oplus H^*(s^{g-2} \Sigma) \oplus \cdots \oplus H^*(s^0 \Sigma)
\]

where \( s^i \Sigma \) is the \( i \)th symmetric product of \( \Sigma \). Here \( H^*(s^i \Sigma) \) is isomorphic to the eigenspace of eigenvalues \( \pm 4(g - 1 - i) \sqrt{-1} \gamma^{-1}, (-1)^{g-1-i} (8, 0) \). The isomorphism respects only \( \mathbb{Z}/2\mathbb{Z} \)-grading and is \( \text{Diff}(\Sigma) \)-equivariant.

Simple computations establish that the dimensions of both vector spaces appearing in Conjecture 24 are the same, i.e. \( g2^g \). The Euler characteristic are both vanishing. Moreover the dimensions of the invariant parts coincide \((\gamma^2 \frac{1}{2})\). Examples 22 and 23 agree with the conjecture.

A deeper reason for the above conjecture is the fact that \( HF^*_\beta \) is the space for a gluing theory of Donaldson invariants associated to the three manifold \( Y = \Sigma \times S^1 \). The gluing theory of Seiberg–Witten invariants should be based on the Seiberg–Witten–Floer homology groups of \( \Sigma \times S^1 \), which are indexed by a line bundle \( L \) (the determinant line bundle of the spin^c-structure on \( Y \)). The only possibilities are \( c_1(L) = \pm 2(g - 1 - i) \text{PD} [S^1] \), \( 0 \leq i \leq g - 1 \) (see [9, Section 6]). It is believed that the Seiberg-Witten-Floer groups for \( L \) are isomorphic to \( H^*(s^i \Sigma) \).

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