Linearized methods
for ordinary differential equations

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Abstract

The conservation properties, singularities and nonlinear dynamics of both time-linearized methods which provide piecewise analytical solutions and keep the independent variable continuous, and implicit, linearized \( \theta \)-techniques which are based on discretization and linearization, are analyzed in this paper. It is shown that both methods are implicit and provide explicit maps, but they do not preserve the energy in conservative systems. It is also shown that time-linearized methods preserve both the fixed points and the linear stability of the original ordinary differential equation, whereas linearized \( \theta \)-techniques do preserve the fixed points and the linear stability of attractors, but the stability of the repellers depends on the time step and the implicitness parameter. The results clearly indicate that the linearized \( \theta \)-techniques which more faithfully reproduce the nonlinear dynamics of the original ordinary differential equation are second-order accurate in time. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

In previous work [1,2], two families of linearized methods for the solution of both autonomous and nonautonomous ordinary differential equations (odes) were developed. In the first family, a \( \theta \)-technique was used for the time
discretization and the resulting nonlinear system of algebraic equations was linearized with respect to the previous time level. As a consequence, a linear system of algebraic equations was obtained [1]. This family of finite difference schemes is here referred to as linearized $\theta$-methods which are linearly implicit techniques of the Rosenbrock type [3].

In the second family of linearized techniques, the time or independent variable was kept continuous and the right-hand side of the odes was approximated by the first two terms of its Taylor series expansion in a piecewise fashion [2]. As a consequence, a system of linear odes was obtained. This system can be solved analytically in each time interval; however, the work required to obtain this analytical solution is large for large systems of odes [2] even if the Schur's normal form theorem is employed to reduce this system to a triangular form. Moreover, the analytical solution involves the exponential of a matrix which, in general, is not nilpotent and, therefore, the Taylor series expansion of the exponential of this matrix contains infinite terms. Hereon, the methods based on the linearization of the right-hand side of the odes are referred to as time-linearized methods.

Both linearized $\theta$-techniques and time-linearized methods can be applied to partial differential equations (pdes) [4,5]. When linearized $\theta$-techniques are employed to solve pdes, the time discretization and linearization result in linear odes or pdes in one- or multi-dimensional problems, respectively, whose coefficients and right-hand sides depend on the local values of the dependent variables at the previous time step, i.e., the resulting pdes are linear but with variable coefficients. The time discretization thus results in Rothe's techniques or methods of lines in space for which there is a large mathematical theory, especially for elliptic equations. The discretization of the spatial operators in Rothe's methods can be achieved by standard second-order accurate finite difference schemes [4], compact, three-point, fourth-order accurate finite difference methods [6], or non-standard finite difference schemes which provide piecewise exponential fittings to local one-dimensional operators [5]. However, the latter techniques suffer from analogous problems to those associated with time-linearized methods for odes, i.e., it is necessary to apply Schur's normal form theorem to the system of locally linear one-dimensional operators to sequentially couple them and, thus, obtain their analytical solutions. On the other hand, the use of time-linearized methods for pdes requires that the spatial coordinates be first discretized and then a linearization of the resulting odes. Therefore, these techniques correspond to methods of lines in time [4].

In this paper, both linearized $\theta$-techniques and time-linearized methods are considered and applied to nonlinear odes. In particular, the singularities of the former, the conservation properties of both techniques, and different approximations to the matrix exponential of the latter are studied. Both techniques are applied to a nonlinear ode which corresponds to the time reversal of the logistic equation.
2. Time-linearized methods

Consider the following autonomous system of nonlinear odes

$$\frac{du}{dt} = f(u),$$

subject to $u(0) = u_0$, where $u \in \mathbb{R}^N$ and $f \in \mathbb{R}^N$.

If the interval of integration $I = (0, T]$ is decomposed into nonoverlapping subintervals $I_n = (t^n, t^{n+1}]$ so that $I = \bigcup I_n$, and Eq. (1) is approximated in each $I_n$ by

$$\frac{dv}{dt} = f(v^n) + J(v^n)(v - v^n),$$

where $v^0(0) = u_0$ and $v$ is an approximation to $u$ in $I_n$, then one can easily obtain

$$v^{n+1} = v^n + \exp(k_n J_n) \int_t^{t^{n+1}} \exp(-(s - t^n)J_n) f(s) \, ds,$$

where $v^n = v(t^n)$, $k_n = t^{n+1} - t^n$, and $J = \partial f / \partial u$ is the Jacobian matrix.

If $J_n$ is nonsingular, then Eq. (3) can be written as

$$v^{n+1} = v^n + (\exp(K_n J_n) - I) J_n^{-1} f_n,$$

where $I$ denotes the identity matrix.

Both Eqs. (3) and (4) involve matrix exponentials whose Taylor series expansions contain infinite terms unless $J_n$ is nilpotent, i.e., there exists an $M$ such that $J_n^M = 0$. This means that approximations must be made on the matrix exponentials in order to approximate $v^{n+1}$. Here, we consider Padé approximants $R^n_S(A) = \sum_0^S (a_i A^i) [\sum_0^T (b_i A^i)]^{-1}$ which are an $(S,T)$ rational approximation of order $p$ to $\exp(A)$ if

$$R^n_T(A) = \exp(A) + O(A^{p+1}).$$

We also consider Padé approximants $Q^n_T(A) = [\sum_0^T (c_i A^i)]^{-1} \sum_0^S (d_i A^i)$. The coefficients of both approximants are determined so that both the approximation error is small and the approximation is acceptable [7,8], i.e., a rational approximation $R(x)$ to $e^x$ is said to be $A$-acceptable if $|R(x)| < 1$ whenever $\Re(x) < 0$, $A_0$-acceptable if $|R(x)| < 1$ whenever $x \in \mathbb{R}$ and $x < 0$, and $L$-acceptable if it is $A$-acceptable and $|R(x)| \to 0$ as $\Re(x) \to -\infty$. In the next sections, Padé approximants are applied to both $v \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

2.1. Linear stability for scalar equations

If $f$ is a linear function of $u \in \mathbb{R}$ then $v = u$; therefore, the linear stability of Eq. (1) coincides with that of Eq. (2). We, therefore, only need to consider the linear stability of
for which time-linearized methods yield
\[ v^{n+1} = v^n e^{ik}, \quad (7) \]
and, therefore, these methods are \( A \)-stable.

The \( R^1_0 = 1 + J_n k \) approximant to the exponential yields (\( p = 1 \))
\[ v^{n+1} = v^n + f_n k, \quad (8) \]
which is the well-known Euler forward method, whereas the \( R^2_0 = 1 + J_n k + \frac{1}{2} J_n^2 k^2 \) approximant to the exponential provides (\( p = 2 \))
\[ v^{n+1} = v^n + f_n (k + \frac{1}{2} J_n^2 k^2), \quad (9) \]
which corresponds to a second-order accurate Taylor’s explicit method. On the other hand, the approximant \( R^1_1 = \frac{1 + J_n k/2}{1 - J_n k/2} \) to the exponential is both \( A \)- and \( L \)-acceptable and yields (\( p = 1 \))
\[ v^{n+1} = v^n + \frac{f_n k}{1 - J_n k/2}, \quad (10) \]
which has the same linear stability characteristics as the standard implicit technique, i.e., it is \( A \)-stable. However, it should be stressed that the time-linearized method results in an explicit map (cf. Eqs. (3) and (4)).

The approximant \( R^1_1 = \frac{1 + J_n k/2}{1 - J_n k/2} \) to the exponential is \( A \)-acceptable but not \( L \)-acceptable and yields (\( p = 1 \))
\[ v^{n+1} = v^n + \frac{f_n k}{1 - J_n k/2}, \quad (11) \]
which has the same linear stability characteristics as the standard trapezoidal rule, i.e., it is \( A \)-stable and not \( L \)-stable, but the time-linearized method is explicit. On the other hand, the approximant \( R^0_2 = 1/(1 - J_n k/2 + J_n^2 k^2/2) \) to the exponential is \( A_0 \)- and \( L \)-acceptable and yields (\( p = 1 \))
\[ v^{n+1} = v^n + f_n k \frac{1 - J_n k/2}{1 - J_n k + J_n^2 k^2/2}, \quad (12) \]
which may not be \( A \)-stable.

2.2. Padé approximants for systems of odes

As stated above, Padé approximants can be used in Eq. (3) or Eq. (4). If they are used in Eq. (3) such approximants may not be integrated analytically. Here, Padé approximants are used in Eq. (4). If the matrix exponential is approximated by \( R^1_1 = (I + \frac{1}{2} kA)(I - \frac{1}{2} kA)^{-1} \), then one can easily obtain
\[ v^{n+1} = v^n + kA(I - \frac{1}{2}kA)^{-1}A^{-1}f_n, \]

which requires the inversion of two matrices. Similarly, the approximant \( R^2_1 = (I + \frac{1}{2}kA + \frac{1}{12}k^2A^2)(I - \frac{1}{2}kA + \frac{1}{12}k^2A^2)^{-1} \) also requires the existence and inversion of two matrices. On the other hand, the approximant \( Q^2_1 = (I - \frac{1}{2}kA)^{-1} \times (I + \frac{1}{2}kA) \) yields

\[ v^{n+1} = v^n + k(I - \frac{1}{2}kA)^{-1}f_n, \]

which only requires an inverse matrix. Therefore, in time-linearized methods, it is more useful to work with \( Q^2_1 \) rational approximations that with \( R^2_1 \) ones.

### 2.3. Error of time-linearized methods

Here, we consider the errors associated with time-linearized techniques. To this end, we define \( E = \|u - v\| \) and subtract Eqs. (1) and (2) to obtain

\[ u - v = \int_{t_{n-1}}^{t_n} (f(u) - f(v)) + J_n(v - v_{n-1}) \, dt, \]

which, for \( f \) satisfying a Lipschitz condition of constant \( L \), can be written as

\[ E \leq LE_n + L \int_{t_{n-1}}^{t_n} E \, dt + K_n \|v - v_n\|_{\text{max}}(t - t_n), \]

where \( K_n = L + \|J_n\| \) and \( t^n \leq t \leq t^{n+1} \).

Eq. (16) can be written as

\[ \frac{dE}{dt} - LE \leq K_n \|v - v_n\|_{\text{max}}, \]

for which Gronwall’s lemma yields

\[ E(t) \leq E_n \exp(Lk) + \frac{K_n}{L} \|v - v_n\|_{\text{max}}(\exp(L(t - t_n)) - 1), \]

which results in

\[ E_n \leq E_0 \exp(nLk) + \frac{\exp(nLk) - 1}{\exp(Lk) - 1}, \]

where \( E_0 = 0 \) and \( P = 2\|v - v_n\|_{\text{max}}(\exp(Lk) - 1) \). Moreover, the analytical solution to Eq. (3) for \( v \in \mathbb{R} \), i.e.,

\[ v - v_n = \frac{f_n}{J_n} (\exp(J_n(t - t_n)) - 1), \]

implies that
\|v - v_n\| \leq \left\| \frac{f_n}{J_n} \right\| \| \exp(J_n(t - t_n)) \| \leq \left\| \frac{f_n}{J_n} \right\| \exp(\|J_n\|k), \tag{21}

\text{and}

\|E_n\| \leq 2 \left\| \frac{f_n}{J_n} \right\| \exp(Lk)(\exp(nLk) - 1), \tag{22}

where \(J_n\) was assumed to be different from zero.

It should be noted that, in general, \(u_n \neq v_n\) and, therefore, the solution of piecewise time-linearized methods deviates from the exact one on account of both the linearization of the right-hand side of the differential equation and the discrepancies between the initial conditions of the exact solution and that of time-linearized techniques at the left ends of each time interval. For linear odes with constant coefficients, time-linearized methods provide the exact solution.

In order to improve the accuracy of \(v_n\), one may use the following iterative procedure which is similar to defect- or deferred-correction methods: the exact solution to Eq. (1) is assumed to be \(u(t) \approx v(t) + e^k(t)\) where \(v(t)\) is the solution obtained from time-linearized methods. When this is substituted into Eq. (1) and the resulting equation is linearized with respect to \(v(t)\), one can obtain the following linear ode for \(e(t)\)

\[ \frac{de^k}{dt} = f(v) - f(v_n) - J(v_n)(v - v_n) - J(v)e^k, \tag{23} \]

which may, in principle, be solved analytically, at least for scalar equations, since \(v(t)\) is known and \(e^0(0) = 0\). This iterative process should be repeated until a prescribed tolerance is reached, i.e., until \(\sup|e^{k+1}(t) - e^{k+1}(t)|\) is equal to or smaller than a prescribed tolerance for \(0 < t \leq k\). This procedure allows one to obtain accurate values of \(v_1\) and can be readily extended to the next time interval. Moreover, a large increase in the efficiency of this iterative method may be achieved by expanding \(f(v)\) and \(J(v)\) in the above equation in Taylor series and integrating the resulting equation analytically or numerically.

### 2.4. Conservation properties

Although time-linearized methods provide piecewise analytical solutions to nonlinear odes, they may be limited because of accuracy considerations. In fact, the error analysis presented above indicates that, although the errors of these methods are bounded, small time steps may be needed if the Lipschitz constant is large as it occurs in stiff problems. Furthermore, although these techniques are exact for linear, constant coefficient odes, they may not be suitable for long-term integration of Hamiltonian systems, molecular dynamics, etc., where energy must be conserved and the symplectic structure of the flow must be preserved. Although it has been shown that it is impossible for a numerical scheme to conserve both the energy and the symplectic structure of
the flow, the time-linearized methods presented here do not, in general, preserve either one as shown next.

For the sake of simplicity, we shall consider the following second-order ode
\[
\frac{d^2x}{dt^2} + \frac{\omega^2}{2} x^2 = 0, \tag{24}
\]
which has the following invariant \((E = \text{constant})\)
\[
E = 3\left(\frac{dx}{dt}\right)^2 + \omega^2 x^3. \tag{25}
\]
If a time-linearized method is employed to solve Eq. (24), then it can be easily shown that the error in \(E\) can be written as
\[
E - E_{n+1} = \frac{3}{8} y_n^2 \sin^2 \theta (\cos \theta - 1) - 3\lambda_n x_n y_n \sin \theta \cos \theta
+ \frac{3}{4\lambda_n} \omega^2 x_n^2 y_n (1 + \cos \theta)^2 \sin \theta + \frac{\omega^2}{\lambda_n^2} y_n^3 \sin^3 \theta
+ \frac{\omega^2}{8} x_n^3 (\cos^2 \theta + 1)(\cos \theta - 1), \tag{26}
\]
where \(y = dx/dt, \theta = \lambda_n k\) and \(\lambda^2 = \omega^2 x_n\).

3. Linearized \(\theta\)-methods

Linearized \(\theta\)-methods are obtained from the application of \(\theta\)-techniques to Eq. (1) and the linearization of the terms at \(t^{n+1}\) in the resulting nonlinear equation with respect to the previous time level, i.e.,
\[
v^{n+1} - v^n = k(f(v^n) + \theta J(v^n)(v^{n+1} - v^n)), \tag{27}
\]
and provide and explicit expression for \(v^{n+1}\), i.e.,
\[
v^{n+1} = v^n + k(I - k\theta J(v^n))^{-1} f(v^n). \tag{28}
\]
These methods are \(A\)-stable, do require the inversion of a matrix, and are second-order accurate for \(\theta = 1/2\) and first-order accurate, otherwise. They are exact for linear odes with constant coefficients. When applied to
\[
\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{29}
\]
yield
\[
E_{n+1} = \frac{E_n}{1 + k^2 \omega^2}, \tag{30}
\]
and \(E_{n+1} = E_n\) for \(\theta = 1\) and \(\frac{1}{2}\), respectively. Therefore, they do conserve energy for linear, conservative, second-order odes with constant coefficients for \(\theta = \frac{1}{2}\).
and decrease the energy for $\theta = 1$. When applied to Eq. (24), these methods yield

$$E_{n+1} - E_n = \frac{\omega^2}{D_3} k_3^3 y_n^3 + \frac{3}{D_1} (-4D_2 + 8D_1 - 4) y_n^2 + \frac{48}{k^2 \omega^2 D_3} (D - 1)^4 y_n^4$$

$$+ \frac{64}{k^6 \omega^6 D_3} (D - 1)^3 (3D_2 - 3D + 1 - D_3),$$

(31)

for $\theta = \frac{1}{2}$, where $D = 1 + k^2 \omega^2 x_n / 4$. Therefore, linearized $\theta$-methods do not, in general, conserve energy.

4. Applications of linearization techniques

In this section, linearized $\theta$-methods and time-linearized techniques are employed to analyze the numerical solution of

$$\frac{du}{dt} = -u(1 - u),$$

(32)

which may be referred to as the time-reversed logistic equation since it can be obtained from the latter by a mirror reflection in time. The fixed points of this equation are $u = 0$ and $u = 1$ which are linearly stable and unstable, respectively.

The exact solution of Eq. (32) is

$$u(t) = \frac{u(0)e^{-t}}{1 - u(0) + u(0)e^{-t}},$$

(33)

which exhibits blow-up at $t = \ln |u(0)/(u(0) - 1)|$ for $u(0) > 1$. This blow-up in finite time may not be captured by the linearized techniques presented in this paper as shown below.

In this paper, we have investigated both analytically and numerically the maps of both time-linearized and implicit, linearized $\theta$-methods for $u_0 \in [-10, 10], \theta = 0.5$ and 1, and $k \in [0.001, 10]$ for the time-reversed logistic equation as indicated in the next sections. An approximate estimate of the larger time steps that could be used in order to obtain accurate discrete approximations to the time-reversed logistic differential equation can be obtained from $k \leq \min |1/(1 - 2u)|$ which, for $u = -10$, implies that $k \leq 0.0476$. The fact that, in some cases, time steps larger than those required to obtain accurate, discrete approximations to the time-reversed logistic differential equation by means of both time-linearized and implicit, linearized $\theta$-methods were used is due to the fact that the maps of both time-linearized and implicit, linearized $\theta$-methods were considered as such without taking into account the time-reversed logistic differential equation to which they were applied and that the time step was selected to find the differences between the
4.1. Time-linearized methods

The discretization of Eq. (32) by means of time-linearized techniques yields

\[ u_{n+1} = u_n + \frac{u_n(1 - u_n)}{1 - 2u_n} \left( \exp((2u_n - 1)k) - 1 \right) \equiv g(u_n; k), \]  

(34)

which for \( u_n = \frac{1}{2} \) yields \( u_{n+1} = (2 - k)/4 \) by using L’Hôpital rule. Therefore, for \( u_n = \frac{1}{2} \) and \( k = 2 \), \( u_{n+1} = 0 \).

The fixed points of Eq. (34) are 0 and 1, and coincide with those of Eq. (32); moreover, \( \frac{dg}{du}(0; k) = e^{-k} < 1 \) and \( \frac{dg}{du}(1; k) = e^k > 1 \) and, therefore, time-linearized methods preserve the linear stability characteristics of the fixed points of Eq. (32). For both \( 1 > u > \frac{1}{2} \) and \( \frac{1}{2} > u > 0 \), \( g(u; k) - u < 0 \) and \( u_{n+1} < u_n \), while, for \( u_n = \frac{1}{2} \), \( u_{n+1} < u_n \). Therefore, all \( 1 > u_n \geq 0 \) are attracted to the fixed point located at \( u = 0 \). For both \( u > 1 \) and \( 0 > u, g(u; k) - u > 0 \) and \( u_{n+1} > u_n \); therefore, \( u_n \) is monotonically increasing for \( u > 1 \), and tends to the fixed point located at \( u = 0 \) for \( u < 0 \). Furthermore, by using L’Hôpital rule, one can easily show that \( \frac{dg}{du}(\frac{1}{2}; k) = 1 - k^2/4 \).

Fig. 1 shows the first 100 iterates associated with \( u_0 = -10 \) and different step sizes, and indicates that the iterates tend monotonically to the stable fixed point \( u = 0 \) at a rate that increases as \( k \) is increased. Similar behavior to that exhibited in Fig. 1 has also been observed for the same values of \( k \) and \(-10 \leq u_0 < 1 \).

For larger values of \( k \), it has been observed that the iterates also converge to \( u = 0 \) for \(-1 \leq u_0 < 1 \) but the convergence is monotonically increasing for

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**Fig. 1.** \( u(n) \) as a function of \( n \) (left) and \( u(n+1) \) as a function of \( u(n) \) (right) for time-linearized methods with \( u(0) = -10 \). (×): \( k = 0.001 \); (○): \( k = 0.01 \); (+): \( k = 0.1 \); (×): \( k = 0.5 \).
$u_0 < 0$ and first increasing and then decreasing for $u_0 > 0$ as illustrated in Figs. 2 and 3, respectively.

Fig. 4 illustrates both the basin of attraction and the Lyapunov exponents for $n = 10^7$ as a function of $u_0$ for different step sizes, and indicates that all $u_0 < 1$ are attracted to the stable fixed point, $u = 0$, yield negative Lyapunov exponents, and the Lyapunov exponents increase in magnitude as $k$ is increased. (The basins of attraction of the fixed points of the map are here defined as the sets of $(u_0, k)$ such that $u_n$ tends to either fixed point as $n \to \infty$.) The only positive Lyapunov exponents correspond to the unstable fixed point $u = 1$. Note that, since time-linearized methods yield monotonically increasing solutions for $u_0 > 1$, no results are presented for these values.
4.2. Linearized \( h \)-methods

The discretization of Eq. (32) by means of (implicit, i.e., \( h \)-linearized) techniques yields

\[
\begin{align*}
  u_{n+1} &= u_n - \frac{k u_n (1 - u_n)}{1 + k \theta (1 - 2u_n)} = g(u_n; k, \theta),
\end{align*}
\]

which shows that these methods yield explicit maps and preserve the fixed points of Eq. (32). Here, we are mainly interested in \( 0 < \theta \leq 1; \theta = \frac{1}{2} \) and 1 correspond to linearized Crank–Nicolson and implicit techniques, respectively, which are \( O(k^2) \) and \( O(k) \) accurate, respectively.

For \( k\theta = 1 \), Eq. (35) yields

\[
\begin{align*}
  u_{n+1} &= u_n \left( 1 - \frac{k}{2} \right)^n,
\end{align*}
\]

which implies that \( |u_n| \to \infty \) in an oscillatory manner for \( k > 2 \), and converges to the fixed point \( u = 0 \) for \( 0 < k < 2 \). Moreover, \( g(u; k, \theta) = 0 \) for \( u = 0 \) and

\[
\begin{align*}
  u = \frac{1 - \theta - 1/k}{1 - 2\theta},
\end{align*}
\]

provided that \( \theta \neq \frac{1}{2} \) and \( k\theta \neq 1 \). The singular points of \( g(u; k, \theta) \) are

\[
\begin{align*}
  u = \frac{1}{2k\theta} (1 + k\theta),
\end{align*}
\]

provided that \( k\theta \neq 1 \). As indicated previously in this section, the time-reversed logistic equation may blow up in finite time if \( u(0) > 1 \).

The linear stability of the fixed points of Eq. (35) depends on the value of \( g'(u; k, \theta) \) where the prime denotes derivative with respect to \( u \). Since \( g'(0; k, \theta) = (1 + k(\theta - 1))/(1 + k\theta) \), implicit linearized \( h \)-methods with

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Fig. 4. Basins of attraction time-linearized methods. \( n = 10^3 \); (\( \ast \)) \( k = 3.5 \); (\( \circ \)) \( k = 5 \); (\( \times \)) \( k = 7 \).
$0 < \theta \leq 1$ preserve the linear stability characteristics of the stable fixed point of the time-reversed logistic equation. For $u = 0$, it can be easily shown that $0 < g'(0; k, \theta) < 1$ requires that $1 > k(1 - \theta)$, and, for these step sizes, the convergence to the fixed point $u = 0$ is monotonic, whereas, for $1 < k(1 - \theta)$, the convergence is oscillatory.

Since $g'(1; k, \theta) = (1 + k(1 - \theta))/(1 - k\theta)$, it can be easily shown that the convergence towards the unstable fixed point $u = 1$ is oscillatory if $2 < k(2\theta - 1)$ which is not satisfied if $\theta = \frac{1}{2}$, while it requires that $k > 2$ for $\theta = 1$; if the step size does not satisfy this inequality, $|g'(1; k, \theta)| > 1$ and the map diverges.

Figs. 5 and 6 illustrate the basins of attraction of linearized $\theta$-methods for $\theta = \frac{1}{2}$ as functions of the initial value of $u$ and the step size $k$ and correspond to $n = 10^7$. These figures clearly indicate that, for $\theta = \frac{1}{2}$, all the initial values tend to the stable fixed point $u = 0$, except for $u_0 = 1$ which remains in this unstable attractor.

The basins of attractions corresponding to $\theta = 1$ are illustrated in Figs. 7 and 8. The results presented in Fig. 7 indicate that, for all $u_0 < 0$, the iterates

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*Fig. 5. Basins of attraction for linearized $\theta$-methods. $n = 10^7; \theta = \frac{1}{2}; (\circ): u = 1; (\ast): u = 0$; the circles for $u_0 = 1$ indicate that this is an unstable fixed point.*
tend to the stable attractor located at $u = 0$, whereas, for $k > 1$ and $u_0 \geq 1$, the iterates tend to the unstable fixed point located at $u = 1$.

For smaller values of $k$ and $u_0$ than those presented in Fig. 7, it is observed that, for $\theta = 1$, and all initial values of $u_0$ except $u_0 = 1$ are attracted to the stable fixed point (Fig. 8). $u_0 = 3$ is also attracted to the unstable fixed point for $k = 0.5$ (Fig. 8). Note that this is not a singular point of linearized $\theta$-methods.

The Lyapunov exponents corresponding to $\theta$-methods with $\theta = \frac{1}{2}$ and 1 are shown in Figs. 9 and 10 and Figs. 11 and 12, respectively; these exponents were calculated with $n = 10^7$. Figs. 9 and 10 indicate that all the Lyapunov exponents are negative for $u_0 \neq 1$ and $\theta = \frac{1}{2}$ on account of the linear stability of the stable fixed to which the iterates are attracted. The positive Lyapunov exponents associated with $u_0 = 1$ are a consequence of the fact that this point is a repeller or unstable fixed point. For $k = 2$ and $\theta = \frac{1}{2}$, Eq. (36) indicates that $u_{n+1} = 0$; therefore, no Lyapunov exponents for these values appear in Fig. 9. For $\theta = 1$, the results presented in Figs. 11 and 12 clearly show that the Lyapunov exponents are negative except for $u_0 = 1$ and small step sizes (Fig. 12). For $|u_0| > 1$ and $k \geq 1$, the results presented in Fig. 11 indicate that
the Lyapunov exponents undergo a sudden change as the initial iterate becomes positive, and are larger for \( u_0 > 0 \) than for \( u_0 < 0 \). The Lyapunov exponent corresponding to \( u_0^3 \) and \( k^0.5 \) is positive for \( h^1 \) as indicated in Fig. 11.

The results presented in Figs. 9–12 indicate that the most accurate linearized \( h \)-methods (which correspond to \( h^1/2 \)) result in maps which tend to the stable fixed point, have smaller Lyapunov exponents than those corresponding to \( h^1 \), and do not exhibit islands of attraction to unstable fixed point such as that corresponding to \( u_0^3, k^0.5 \) and \( h^1/2 \). These figures also indicate that the magnitude of the Lyapunov exponents increase as the step size is decreased. For \( k^0.01 \), the Lyapunov exponents were found to depend on \( n \) even for \( n = 10^7 \).

Although not shown here, the first iterations of implicit, linearized \( \theta \)-methods were found to be oscillatory for \( k \leq 3 \) and \( \theta = \frac{1}{3} \), in agreement with the analytical considerations discussed above. On the other hand, linearized \( \theta \)-methods with \( \theta = 1 \) have a larger monotonic behavior towards the fixed points than those with \( \theta = \frac{1}{2} \).
5. Conclusions

Time-linearized and implicit, linearized \( \theta \)-methods for the nonlinear odes have been developed and applied to the time-reversed logistic differential equation. Time-linearized techniques are based on the linearization of the right-hand side of the odes and provide piecewise analytical solutions which depend on matrix exponentials. Although, the system of linear odes resulting from the linearization may be transformed by means of the Schur normal form theorem into a triangular one which may be solved in a sequential manner, the work required by this transformation is too costly, but can be substantially reduced by approximating the matrix exponentials by means of Padé approximants which may require the inversion of one or two matrices at each time level.

In order to increase the accuracy of time-linearized techniques whose errors have been bounded, an iterative predictor–corrector technique has been proposed. This technique is based on the discrepancies between the exact and numerical values at each time level, and is based on the time linearization at two successive time levels and a defect-correction argument.

Fig. 8. Basins of attraction for linearized \( \theta \)-methods. \( n = 10^7; \theta = 1 \); (\( * \)): \( u = 1 \); (\( * \)): \( u = 0 \); the circles for \( u_0 = 1 \) indicate that this is an unstable fixed point.
Fig. 9. Lyapunov exponents. Top: (•) $k = 10$; (○) $k = 9$; (+) $k = 8$; (×) $k = 7$. Middle: (•) $k = 6$; (○) $k = 5$; (+) $k = 4$; (×) $k = 3$. Bottom: (○) $k = 1$. $n = 10^7$; $\theta = 0.5$. 
Fig. 10. Lyapunov exponents. Top: (*) $\leftrightarrow k = 0.9$; (o) $\leftrightarrow k = 0.8$; (+) $\leftrightarrow k = 0.7$; (x) $\leftrightarrow k = 0.6$.
Middle: (*) $\leftrightarrow k = 0.5$; (o) $\leftrightarrow k = 0.4$; (+) $\leftrightarrow k = 0.3$; (x) $\leftrightarrow k = 0.2$.
Bottom: (*) $\leftrightarrow k = 0.1$; (o) $\leftrightarrow k = 0.01$. $n = 10^3$; $\theta = 0.5$. 
Fig. 11. Lyapunov exponents. Top: $(*) \leftrightarrow k = 10; (\circ) \leftrightarrow k = 9; (+) \leftrightarrow k = 8; (\times) \leftrightarrow k = 7$. Middle: $(*) \leftrightarrow k = 6; (\circ) \leftrightarrow k = 5; (+) \leftrightarrow k = 4; (\times) \leftrightarrow k = 3$. Bottom: $(*) \leftrightarrow k = 2; (\circ) \leftrightarrow k = 1. n = 10^7; \theta = 1.
Fig. 12. Lyapunov exponents. Top: (*) $\leftrightarrow k = 0.9$; (o) $\leftrightarrow k = 0.8$; (+) $\leftrightarrow k = 0.7$; (x) $\leftrightarrow k = 0.6$.
Middle:  (*) $\leftrightarrow k = 0.5$; (o) $\leftrightarrow k = 0.4$; (+) $\leftrightarrow k = 0.3$; (x) $\leftrightarrow k = 0.2$. Bottom:  (*) $\leftrightarrow k = 0.1$; (o) $\leftrightarrow k = 0.01$. $n = 10^3$; $\theta = 1$. 

Implicit, linearized $\theta$-methods are linearly implicit techniques that require the inversion of a matrix at each time level but they are based on the discretization and linearization of the original odes; therefore, the independent variable is discrete, whereas this variable is continuous in time-linearized schemes. Despite their implicit character, both time-linearized and linearized $\theta$-methods provide explicit maps.

It has been shown that time-linearized methods are valid even when the Jacobian matrix is singular but they do not preserve energy in conservative systems. Implicit, linearized $\theta$-methods do not preserve energy either, may have singular points and spurious zeros, and are dynamically richer than time-linearized techniques because their maps depend on both the step size and the implicitness parameter. For the time-reversed logistic equation, it was found that time-linearized methods preserve both the fixed points and the linear stability of the fixed points regardless of the step size, whereas linearized $\theta$-methods do preserve the fixed points and the stability of the attractors; the stability of the repellers may depend on both the step size and the implicitness parameters. It has also been found that implicit linearized $\theta$-methods may not capture blow-up phenomenon in finite time, and that the techniques which more faithfully reproduce the nonlinear dynamics of odes are those associated with a Crank–Nicolson discretization. First-order accurate, implicit, linearized $\theta$-methods may predict that iterates are attracted to the unstable fixed point for some time steps and to the stable one for others.

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References

