Torus rational fibrations

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Abstract

We study rational fibrations where the fibre is an $r$-dimensional torus and the base is a formal space. We make use of the Eilenberg–Moore Spectral Sequence to prove the Toral Rank Conjecture in some cases. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The purpose of this note is to present a class of manifolds for which the Toral Rank Conjecture holds. Recall that for a finite-dimensional connected smooth manifold $E$ we call rank of $E$, and denote it by $rk(E)$, the maximum integer $r$ such that there is an almost free action of the $r$-dimensional torus $\mathbb{T}^r$ on $E$ (see [4, Ch. 5; 8]). Then the Toral Rank Conjecture is the following

Conjecture 1 (Felix [4, Section 5.2]). Let $E$ be a finite-dimensional smooth simply connected manifold and let $r = rk(E)$. Then the (rational) cohomology of $E$ has dimension at least $2^r$.

Recall that any connected CW-complex $M$ of finite type has a (minimal) Sullivan model $(\Lambda X_M, d)$ which computes its rational cohomology, $H^*(\Lambda X_M, d) = H^*(M)$ (when $M$ is simply connected, $X_M$ gives also the homotopy of $M$, see [1]). Then we define rational fibration as in [7].
Definition 1. A rational fibration is a couple of maps $T \xrightarrow{i} E \xrightarrow{p} B$ between connected spaces with

- $p \circ i$ homotopically trivial,
- if we consider the KS model of $p$ and the induced map $\psi$,

$$
\begin{align*}
AX_B & \longrightarrow AX_B \otimes AV \longrightarrow AV \\
\| & \quad \| \cong \quad \| \\
AX_B & \longrightarrow AX_E \longrightarrow AX_T
\end{align*}
$$

then $\psi$ is a quasi-isomorphism.

Morally, $T \rightarrow E \rightarrow B$ is a rational fibration if it has a KS model. We remark that if $T \rightarrow E \rightarrow B$ is a fibration with $B$ 1-connected, then it is a rational fibration [5, Section 6]. We shall henceforth assume that $B$ is always 1-connected.

Now, suppose that $T = T'$ acts almost freely on $E$. Then $B = E/T$ is a finite CW-complex and $T \rightarrow E \rightarrow B$ turns out to be a rational fibration [1, Section 5]. This allows us to express conjecture 1 in more natural homotopy terms as

Conjecture 2 (Halperin [8, Problem 1.4]). Let $T \rightarrow E \rightarrow B$ be a rational fibration of finite connected CW-complexes with $B$ 1-connected, in which $T = T'$. Then the rational cohomology of $E$ has dimension at least $2^r$.

One might say that Conjecture 1 is the geometric version and Conjecture 2 is the rational homotopy version. Conjecture 2 implies Conjecture 1 but there is no reason for the converse to hold. The Toral Rank Conjecture 1 is proved in many cases, for example when $E$ is a product of spheres, a homogeneous space or a homology Kähler manifold (see [4, Ch. 5]). Let us state our main two results.

Definition 2. For any finite CW-complex $B$ define $\chi_{\text{even}}(B) = \sum_{i \geq 0} (-1)^i \dim H^{2i}(B)$ and $\chi_{\text{odd}}(B) = \sum_{i \geq 0} (-1)^i \dim H^{2i+1}(B)$.

Theorem 3. Suppose $B$ is formal. If either $\chi_{\text{even}}(B) \neq 0$ or $\chi_{\text{odd}}(B) \neq 0$, then Conjecture 2 is true for $T \rightarrow E \rightarrow B$.

Theorem 4. Suppose $B$ is formal. Write $H^{\text{even}}(B) = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$ for the even dimensional part of the (rational) cohomology algebra of $B$. Then $m \geq n$. If either $m = n$ or $m = n + 1$ then Conjecture 2 holds for $T \rightarrow E \rightarrow B$.

Theorem 4 is a consequence of Propositions 10 and 12 together with Lemma 8. The paper is organised as follows. In Section 2 we give a suitable model for $E$ when $B$ is formal and $T \rightarrow E \rightarrow B$ is a rational fibration. We use it to prove Theorem 3. In Section 3 we recall the Eilenberg–Moore Spectral Sequence and use it to prove Theorem 4. We will assume throughout that all spaces are connected, of finite type and with finite-dimensional rational cohomology. Basic references for rational homotopy theory and Sullivan models are [4, 3, 10], rational fibrations are introduced in [7].
2. A suitable model for $E$

Fix a rational fibration $T \to E \to B$ with $T = T'$. The minimal model of $T$ is $(AX_T, 0)$, where $AX_T = A(y_1, \ldots, y_r)$, $|y_i| = 1$, $1 \leq i \leq r$. Let $(AX_B, d)$ be the minimal model of $B$. By the definition of rational fibration, the KS-extension corresponding to $T \to E \to B$ is

$$(AX_B, d) \to (AX_B \otimes AX_T, D) \to (AX_T, 0),$$

where $(AX_B \otimes AX_T, D)$ is a model (not minimal in general) of $E$. The KS-extension (1) is determined by

$$Dy_i = x_i \in (AX_B)_{2}.$$

Now let $R = \mathbb{Q}[z_1, \ldots, z_r]$ with $|z_i| = 2$, $1 \leq i \leq r$. The algebra morphism $R \to H^*(AX_B)$, $z_i \mapsto x_i$ makes $H^*(B) = H^*(AX_B)$ into an $R$-graded module. Geometrically, this corresponds to the following. As $B$ is 1-connected, the rational fibration $T \to E \to B$ is determined by a (rational) classifying map $B \to BT$, where $BT$ is the classifying space for the torus $T$. This gives a morphism of rings $R = H^*(BT) \to H^*(B)$, which is the one defined above.

**Lemma 5.** Suppose $B$ is formal. Then a model of $E$ is given by $(H^*(B) \otimes H^*(T), d)$, $d(h \otimes y_i) = x_i \cdot h \otimes 1$. In particular, $H^*(E) = H(H^*(B) \otimes H^*(T), d)$.

**Proof.** Consider the model $(AX_B \otimes AX_T, D)$ of $E$ given by the KS-extension (1). As $B$ is formal, there is a quasi-isomorphism $\psi : (AX_B, d) \cong (H^*(B), 0)$. Then $id \otimes \psi : (AX_B \otimes AX_T, D) \to (H^*(B) \otimes AX_T, \hat{D})$ is also a quasi-isomorphism, where $\hat{D} = d$. As $AX_T = H^*(T)$, this means that $(H^*(B) \otimes H^*(T), d)$ is a model of $E$. \hfill $\square$

For any graded $R$-module $M$ we have defined a differential complex $(M \otimes H^*(T), d)$, $d(m \otimes y_i) = x_i \cdot m \otimes 1$. In general, we can ask whether $\dim(M \otimes H^*(T), d) \geq 2^r$ for any finite-dimensional $R$-module $M$. This would give an affirmative answer to Conjecture 2 for any formal space $B$.

Note that for an $R$-module $M$, we have $M = M^{even} \oplus M^{odd}$ and then $(M \otimes H^*(T), d) = (M^{even} \otimes H^*(T), d) \oplus (M^{odd} \otimes H^*(T), d)$.

**Remark 6.** Suppose $B$ is 1-connected. Then the Serre Spectral Sequence for $T \to E \to B$ is the same as the spectral sequence obtained by filtering $AX_B \otimes AX_T$ with $\mathcal{F}^p = (AX_B)^{\leq p} \otimes AX_T$, from the term $E_2$ onwards (see [5]). For this spectral sequence, $E_2^{*,*} = H^*(B) \otimes H^*(T)$ and $d_2$ is the differential $d$ given in Lemma 5. $E_{\infty}$ is isomorphic to the cohomology of $E$ (as vector spaces), so when $B$ is formal $E_3 = E_{\infty}$ and the Serre Spectral Sequence collapses at the third stage.

**Remark 7.** In general, for a rational fibration $T \to E \to B$ with $B$ 1-connected, finiteness of $H^*(B)$ implies the convergence of the Serre Spectral Sequence at a finite
stage. Lemma 5 guarantees convergence at the third stage under the condition of the formality of $B$. To see that this condition is necessary, take for instance $T = \mathbb{T}^2$, $B$ to have minimal model $A_{i=1}^2 B = A(x_1, x_2, u_1, u_2) \otimes AW^{\geq 5}$, where $|x_i| = 2$, $d x_i = 0$, for $i = 1, 2$, $d u_1 = x_i^2$, $d u_2 = x_1 x_2$, and $W$ and $d$ on $W$ are defined in such a way that $H^{\geq 6}(B) = 0$. Then there is a non-zero homology class $[z] \in H^5(B)$, $z = x_2 u_1 - x_1 u_2$.

Put $M = H^*(B) = M^\text{even} \oplus M^\text{odd}$, where

$$M^\text{even} = \mathbb{Q}(1, x_1, x_2, x_2^2), \quad M^\text{odd} = \mathbb{Q}(z).$$

Then $0 \neq [z] \in H(M^\text{odd} \otimes AX_T, d) \subset H(M \otimes AX_T, d)$, but the following computation:

$$d(y_1 y_2 x_1) = x_1^2 y_2 - x_1 x_2 y_1 = (du_1) y_2 - (du_2) y_1 = d(u_1 y_2 - u_2 y_1) + z$$

shows that $0 = [z] \in H^*(E)$. This implies $H^*(E) \neq H^*(B) \otimes H^*(T), d)$ and the Serre Spectral Sequence does not collapse at $E^*_2$.

**Proof of Theorem 3.** Put $M = H^*(B)$. Lemma 5 tells us that the cohomology of $E$ is $H^*(E) = H(M \otimes AX_T, d)$. As above, we write $M = M^\text{even} \oplus M^\text{odd}$ so that $H^*(E) = H(M^\text{even} \otimes AX_T, d) \oplus H(M^\text{odd} \otimes AX_T, d)$. We are going to check that if $\chi_{\text{even}}(B) \neq 0$ then $\dim H(AX_T \otimes M^\text{even}, d) \geq 2^r$ (the other case being analogous). So we can suppose that $M = M^\text{even}$. Give $V = AX_T \otimes M$ the following bigradation: $V^{k,l} = (AX_T)^{k-l} \otimes M^{2l}$, $k, l \in \mathbb{Z}$. Then $d$ has bidegree $(0, 1)$, so it restricts to $V^{k,*}$. The Euler characteristic of $V^{k,*}$ is $\chi(V^{k,*}) = \sum (-1)^j \binom{k-j}{r} \dim M^{2l}$, so

$$\dim H^*(E) = \dim H(V, d) = \sum_k \dim H(V^{k,*}, d) \geq \sum_k |\chi(H(V^{k,*}, d))|$$

$$= \sum_k |\chi(V^{k,*})| \geq \left| \sum_{k,j} (-1)^j \binom{r}{k-j} \dim M^{2l} \right|$$

$$= \sum_l (-1)^j \dim M^{2l} \sum_{k \in \mathbb{Z}} \binom{r}{k-l} = 2^r |\chi_{\text{even}}(B)| \geq 2^r. \quad \square$$

This theorem covers many examples. For instance, let us recall Example 3 in [4, Section 5.3]. Consider

$$B = \underbrace{\mathbb{C} \mathbb{P}^2 \cdots \mathbb{C} \mathbb{P}^2}_{n}, \quad f_i : B \to \mathbb{C} \mathbb{P}^2$$

given by contracting every $\mathbb{C} \mathbb{P}^2$ expect the $i$th one. Then pull back the universal fibration $T^n = (\mathbb{S}^1)^n \to ET^n \to (\mathbb{C} \mathbb{P}^\infty)^n$ under the map $f = f_1 \times \cdots \times f_n : B \to (\mathbb{C} \mathbb{P}^\infty)^n \to (\mathbb{C} \mathbb{P}^\infty)^n$ to get a rational fibration $T \to E \to B$, with $T = \mathbb{T}^n$. As $B$ is formal and $\chi_{\text{even}}(B) = 2 - n$, Conjecture 2 holds for these fibrations when $n \neq 2$. The case $n = 2$ can be worked out explicitly. In this case, $H^*(B) = \mathbb{Q}(x_1, x_2)/(x_1 x_2, x_1^2 - x_2^2)$, with $|x_1| = |x_2| = 2$. Then the $E_2$ term of the Serre Spectral Sequence of Remark 6 is
(the numbers denote the dimensions)

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so \( \dim E_{\infty}^{00} = 1, \dim E_{\infty}^{21} = 2, \dim E_{\infty}^{42} = 1 \). As \( E_{\infty} = E_3 \), we have \( \dim H^*(E) \geq 4 = 2^n \) (actually we do have equality).

3. Use of Eilenberg–Moore Spectral Sequence

Let \( T \to E \to B \) be a rational fibration with \( T = \mathbb{T}^r \), whose associated KS-extension is (1). Consider the Koszul resolution of \( Q \) given by

\[
K^* = R \otimes AX_T = \mathbb{Q}[z_1, \ldots, z_r] \otimes \Lambda(y_1, \ldots, y_r),
\]

\[
dy_i = z_i, \quad |y_i| = 1, \quad |z_i| = 2, \quad 1 \leq i \leq r.
\]

Filter the model of \( E \) given by \((AX_B \otimes AX_T, D)\) with \( \mathcal{F}^p = AX_B \otimes \mathcal{Z}^p X_T \). Then we get a spectral sequence with

\[
E_2^{**} = H(H^*(B) \otimes AX_T, d) = H(H^*(B) \otimes_R K^*, \tilde{D}) = \text{Tor}_R^*(H^*(B), \mathbb{Q}),
\]

\[
E_\infty^{**} = H^*(E) = H(AX_B \otimes AX_T, D) = H(AX_B \otimes_R K^*, D) = \text{Tor}_R^*(AX_B, \mathbb{Q}).
\]

Again, by Lemma 5, if \( B \) is formal \( E_\infty^{**} \) degenerates at the second stage, i.e. \( H^*(E) = \text{Tor}_R^*(H^*(B), \mathbb{Q}) \). To understand this spectral sequence, consider \((R \otimes AX_B \otimes AX_T, \mathcal{D})\), \( \mathcal{D}|_{X_B} = d, \mathcal{D}|_{z_i} = 0, \mathcal{D}|_{y_i} = 1 \otimes x_i - z_i \otimes 1. \) Then

\[
(AX_B, d) \cong (R \otimes AX_B \otimes AX_T, \mathcal{D}) \cong (AX_B, d) \otimes (R \otimes AX_T, d)
\]

is a quasi-isomorphism. So we have a KS-extension

\[
(R, 0) \to (R \otimes AX_B \otimes AX_T, \mathcal{D}) \to (AX_B \otimes AX_T, D),
\]

where the term in the middle is a model for \( B \) and the term in the right a model for \( E \). Then \( E_\infty^{**} \) is the usual Eilenberg–Moore Spectral Sequence associated to (3).

Geometrically, this corresponds to the following. Suppose \( B \) is 1-connected then the fibration \( T \to E \to B \) is determined by a (rational) classifying map \( B \to BT \) which yields a rational fibration \( E \to B \to BT \) with KS-extension (3) (recall that \( (R, 0) \) is a minimal model for \( BT \)). The Eilenberg–Moore Spectral Sequence associated to this fibration is \( E_\infty^{**} \) (see [9]).

With this understood, we aim to prove Theorem 4. First a technical lemma.
Lemma 8. Let $S = \mathbb{Q}[t_1, \ldots, t_n]$ be a polynomial ring, $m = (t_1, \ldots, t_n)$ maximal ideal, and $f_1, \ldots, f_m \in m$ non-zero elements such that for $I = (f_1, \ldots, f_m)$, $S/I$ is finite dimensional. Then $m \geq n$. If $m = n$, $f_1, \ldots, f_n$ form a regular sequence for $S$. If $m > n$ then we can choose $g_1, \ldots, g_m$ generators of $I$ such that $g_1, \ldots, g_n$ are a regular sequence for $S$.

Proof. Let $S_0$ be the localization of $S$ at $m$. Its Krull dimension is $\text{Kd}(S_0) = n$, so by [2, Theorem 11.14], $m \geq n$. Now suppose $m = n$. Since for any local noetherian ring $A$ and $f \in m$, it is $\text{Kd}(A) - 1 \leq \text{Kd}(A/f) \leq \text{Kd}(A)$, we must have $\text{Kd}(S_i) = n - i$, where $S_i = S_0/(f_1, \ldots, f_i)$, $1 \leq i \leq n$. To prove that $f_1, \ldots, f_n$ is a regular sequence we have to prove that $f_{i+1}$ is not a zero divisor in $S_i$, $0 \leq i \leq n - 1$. Suppose $f_{i+1}$ is a zero divisor. Then there must be a minimal prime $p \varsupseteq (f_1, \ldots, f_i)$ with $f_{i+1} \in p$. By [2, Corollary 11.16], $\text{ht} p \leq i$, so $\text{Kd}(S/p) \geq n - i$, hence $\text{Kd}(S_{i+1}) \geq n - i$, which is a contradiction.

Now suppose $m > n$. We shall construct $g_1, \ldots, g_m$ inductively such that they are a regular sequence and (up to reordering $f_i$) $I = (g_1, \ldots, g_{i-1}, f_i, \ldots, f_m)$. Let $g_1 = f_1$. Suppose $g_1, \ldots, g_{i-1}$ constructed. Then $\text{Kd}(S_0/(g_1, \ldots, g_{i-1})) = n - i + 1$. Let $p_1, \ldots, p_k$ be the minimal primes containing $(g_1, \ldots, g_{i-1})$. Define

$$H_j = \{ \mu = (\mu_1, \ldots, \mu_{m-i+1})/\mu_1 f_1 + \cdots + \mu_{m-i+1} f_m \in p_j \} \subset \mathbb{Q}^{m-i+1},$$

for $j = 1, \ldots, k$. As $i \leq n$, $\text{Kd}(S_0/p_j) \neq 0$, so $H_j$ is a proper linear subvariety of $\mathbb{Q}^{m-i+1}$. As a conclusion, there is an element $\mu$ not lying in any $H_j$, so $g_i = \mu_1 f_1 + \cdots + \mu_{m-i+1} f_m \notin \bigcup p_j$. This means that $g_i$ is not a zero divisor in $S_0/(g_1, \ldots, g_{i-1})$. We reorder $f_1, \ldots, f_m$ suitably and repeat the process. □

Remark 9. The elements $g_i$ obtained in the proof of the previous lemma are not homogeneous in general, even when the elements $f_i$ are so. It is probably the case that we cannot arrange them to be homogeneous.

Proposition 10. Let $B$ be formal and with finite-dimensional cohomology. Suppose that $H^\text{even}(B) = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_n)$. Then Conjecture 2 holds for $T \rightarrow E \rightarrow B$.

Proof. By the discussion above, we only need to bound below the dimension of $\text{Tor}_p(H^\ast(B), \mathbb{Q})$. As in the proof of Theorem 3, this splits as $\text{Tor}_p(H^\text{even}(B), \mathbb{Q}) \oplus \text{Tor}_p(H^\text{odd}(B), \mathbb{Q})$, so it suffices to prove $\dim \text{Tor}_p(H^\text{even}(B), \mathbb{Q}) \geq 2^r$. Put $M = H^\text{even}(B)$. As $M$ is an $R = \mathbb{Q}[z_1, \ldots, z_r]$-algebra, we can suppose that

$$M = \mathbb{Q}[z_1, \ldots, z_r, t_{r+1}, \ldots, t_{r+k}]/(f_1, \ldots, f_{r+k}),$$

where $k \geq 0$ (it is possible that we have added some algebra generator $z_j$ together with a relation $f_j = z_j$, but still we have the same number of generators and relations). To compute $\text{Tor}_p(M, \mathbb{Q})$ this time we will resolve $M$. By Lemma 8, $f_1, \ldots, f_{r+k}$ is a regular sequence for the polynomial ring $S = \mathbb{Q}[z_1, \ldots, z_r, t_{r+1}, \ldots, t_{r+k}]$. Then the
Koszul complex, given by \((S \otimes A(e_1, \ldots, e_{r+k}), d)\), \(d e_i = f_i, |e_i| = |f_i| - 1\), is a free \(S\)-resolution of \(M\). Now, we distinguish between the two cases:

1. If \(k = 0\), the Koszul complex is a free \(R\)-resolution and then \(\text{Tor}_R(M, \mathbb{Q}) = H((R \otimes A(e_1, \ldots, e_r)) \otimes_R \mathbb{Q}, d \otimes_R \mathbb{Q}) = A(e_1, \ldots, e_r)\) has dimension 2'.

2. If \(k > 0\), the same argument yields that \(\text{Tor}_R(M, \mathbb{Q})\) has dimension \(2^{r+k}\). Now, \(S = R \otimes T\), where \(T = \mathbb{Q}[t_{r+1}, \ldots, t_{r+k}]\). There is a spectral sequence with \(E_2 = \text{Tor}_T(\text{Tor}_R(M, \mathbb{Q}), \mathbb{Q})\) converging to \(\text{Tor}_S(M, \mathbb{Q})\). This is given as follows: resolve \(\mathbb{Q}\) as \(R\)-module \(K_R \cong \mathbb{Q}\) and as \(T\)-module \(K_T \cong \mathbb{Q}\). Then \(K_R \otimes K_T \cong \mathbb{Q}\) is an \(S\)-resolution of \(\mathbb{Q}\). The spectral sequence is obtained from

\[
M \otimes_{R \otimes T} (K_R \otimes K_T) = M \otimes_{R \otimes T} ((K_R \otimes T) \otimes_T K_T)
\]

We conclude \(\dim \text{Tor}_T(\text{Tor}_R(M, \mathbb{Q}), \mathbb{Q}) \geq 2^{r+k}\). But \(\dim \text{Tor}_T(N, \mathbb{Q}) \leq 2^k \dim N\), for any finite-dimensional \(T\)-module \(N\). Thus \(\dim \text{Tor}_R(M, \mathbb{Q}) \geq 2^r\). \(\square\)

Remark 11. By a result of Halperin [6] (see also [4, Section 2.6]), if \(B\) is a formal 1-connected rational space with \(H^*(B) = H^{even} \mathbb{Q}_B\), then it has finite-dimensional rational homotopy and \(\gamma(B) = 0\). Many properties are known of these elliptic spaces. Note for instance that such an algebra is always a Poincaré duality algebra. However Proposition 10 is valid also for spaces \(B\) with some odd dimensional cohomology.

Proposition 12. Let \(B\) be formal and with finite-dimensional cohomology. Suppose that \(H^{even} \mathbb{Q}_B = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_n)\). Then Conjecture 2 holds for \(T \rightarrow E \rightarrow B\).

Proof. Again we want to prove that \(\dim \text{Tor}_R(M, \mathbb{Q}) \geq 2^r\), with \(M = H^{even} \mathbb{Q}_B\). Write \(M = \mathbb{Q}[z_1, \ldots, z_r, t_{r+1}, \ldots, t_{r+k}]/(f_1, \ldots, f_{r+k+1})\), \(k \geq 0\), as in the proof of Proposition 10. Suppose first that \(f_1, \ldots, f_{r+k}\) is a regular sequence for \(S = \mathbb{Q}[z_1, \ldots, z_r, t_{r+1}, \ldots, t_{r+k}]\). Put \(\tilde{M} = S/(f_1, \ldots, f_{r+k})\) and \(f = f_{r+k+1}\), so that \(M = \tilde{M}/f \tilde{M}\). The proof of Proposition 10 ensures us that \(\dim \text{Tor}_S(\tilde{M}, \mathbb{Q}) = 2^{r+k}\). Let us consider the two cases separately:

1. If \(k = 0\), take the Koszul complex for the given presentation of \(M\), i.e. \(L^\ast = R \otimes A(e_1, \ldots, e_r, \cdot) \rightarrow M, d e_i = f_i, |e_i| = |f_i| - 1\). The main point is that this is not a resolution (i.e. it is not a quasi-isomorphism). In fact, \(\tilde{L}^\ast = R \otimes A(e_1, \ldots, e_r)\) is an \(R\)-resolution of \(\tilde{M}\) and \(L^\ast = \tilde{L}^\ast \otimes A(e)\) where \(e = e_{r+1}\), \(d e = f\). Filter \(L^\ast\) by powers of \(e\). So we get an spectral sequence with \(E_1^{r,s} = \tilde{M} \otimes A(e)\) and there is only one non-trivial differential \(\tilde{M} \otimes e \rightarrow \tilde{M} \otimes 1, m \otimes e \mapsto f \cdot m \otimes 1\). Then

\[
E_\infty^{r,s} = (\tilde{M}/f \tilde{M} \otimes 1) \oplus (\text{Ann}_\tilde{M}(f) \otimes e)
\]

By Remark 11, \(\tilde{M}\) is a Poincaré duality space. This implies that \(\tilde{M}/f \tilde{M} \otimes \text{Ann}_\tilde{M}(f) \rightarrow \mathbb{Q}\) is a perfect pairing, so it gives an isomorphism \(\text{Ann}_\tilde{M}(f) \cong (\tilde{M}/f \tilde{M})^\vee = M^\vee\). Thus \(H^\ast(L) \cong M \oplus M^\vee\).
Now consider the standard Koszul resolution $K^* \cong Q$. The bicomplex $L^* \otimes_R K^*$ gives two spectral sequences, $E_2^{**}$ and $\hat{E}_\infty^{**}$:

$$E_2^{**} = \hat{E}_\infty^{**} = L^* \otimes_R Q = A(e_1, \ldots, e_{r+1}),$$

$$E_2^{**} = \text{Tor}_R^*(H^*(L), Q) = \text{Tor}_R^*(M, Q) \oplus \text{Tor}_R^*(M, Q)^\vee.$$

$\hat{E}_\infty$ has dimension $2^{r+1}$ and we know that $E_\infty = \hat{E}_\infty$ (as vector spaces), so as $E_2$ converges to $E_\infty$,

$$2 \dim \text{Tor}_R^*(M, Q) = \dim E_2 \geq 2^{r+1},$$

whence the result.

2. If $k > 0$, the same argument yields that $\dim \text{Tor}_R^*(M, Q) \geq 2^{r+k}$. Now, we use the argument in the second case of Proposition 10 to get $\dim \text{Tor}_R^*(M, Q) \geq 2^r$.

In the general case, Lemma 8 ensures us that we can write $M = S/(g_1, \ldots, g_{r+k+1})$ where $g_1, \ldots, g_{r+k}$ form a regular sequence (these elements $g_i$ are non-homogeneous in general). We can use the same argument that we have used above, but this time forgetting the degree, i.e. we consider $S$ concentrated in degree 0 and $|e_i| = -1$, $1 \leq i \leq r+k+1$. Also the Koszul resolution $K^* \cong Q$ has to be graded accordingly. This does not affect to the computation of the dimension of $\text{Tor}_R^*(M, Q)$ although it gives a completely different grading. \qed

Remark 13. Let $B$ be a formal 1-connected rational space whose cohomology is $H^*(B) = H^\text{even}(B) = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_{n+1})$. Then $B$ is always hyperbolic (i.e. it has infinite-dimensional rational homotopy). In fact, since $f_1, \ldots, f_{n+1}$ is not a regular sequence, there is a non-trivial relation $a_1 f_1 + \cdots + a_n f_{n+1} = 0$. Take one of minimal degree. In the bigraded model of $H^*(B)$, $Z_0 = < t_1, \ldots, t_n >$, $Z_1 = < u_1, \ldots, u_{n+1} >$, $du_i = f_i$ and then $a_1 u_1 + \cdots + a_{n+1} u_{n+1} = dv$, for some non-zero $v \in Z_2$. So $Z_2 \neq 0$, which implies the hyperbolicity of $B$ (see [4, Section 7.4]). The author wants to thank Greg Lupton for pointing out this to him.

One can hope of proving Conjecture 2 for $T \to E \to B$, where $H^\text{even}(B) = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_{n+s})$, inductively on $s$, but the argument above does not seem to generalise.

Remark 14. Let $T \to E \to B$ be a rational fibration with $T = \mathbb{P}^r$, but this time we will not suppose that $E$ and $B$ are finite CW-complexes but only finite type CW-complexes. Let $a$ stand for the Krull dimension of $H^\text{even}(B)$. Then the arguments of this section carry out to prove that $\dim H^*(E) \geq 2^{r-a}$ whenever $B$ is formal with $H^\text{even}(B) = \mathbb{Q}[t_1, \ldots, t_n]/(f_1, \ldots, f_m)$, $m = n - a$, $n - a + 1$.

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