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ON ONE NUMERICAL METHOD FOR SOLVING SOME SELF-SIMILARITY PROBLEMS OF GAS-DYNAMICS ON A MULTIPROCESSOR

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In this paper we consider mathematical models of some problems of natural science, for example, self-similarity problems of gas-dynamics giving rise to boundary problems of first order ordinary differential equations (ODE) with one parameter. The boundary problems of first order ODE with one parameter are considered in \cite{1, 2}, where iterative methods based on the implementation of Newton's Method, are presented. Next, an iterative method for solving the boundary value problem of the first order system of ODE with one parameter on a multiprocessor type SIMD\cite{3} is shown. The convergence of this process is proved and the speed of convergence is estimated. The feasibility of this method is illustrated for the one dimensional instability movement of gas arising from the movement of the piston in presence of a volume source (volume channel) of mass, impulse and energy in gas. Finally the results are given.

Keywords: Newton's method; Cauchy's problem; Chord's method; Interpolation formulae


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1. THE MOVEMENT OF THE PISTON WITH ACCELERATION AND DECELERATION IN AN ENVIRONMENT WITH SOURCES

Let us consider the one-dimensional instability movement of gas arising from the movement of the piston in presence of volume sources (volume channel) of mass, impulse and energy in gas. Let us suppose that gas is calm and has been distributed in space density at a certain moment in time. The corresponding mathematical model is as follows:

\[
\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial r} + \rho \frac{\partial v}{\partial r} = A T^{n_1} \rho^{n_2}, \quad \rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} \right) + RT \frac{\partial \rho}{\partial r} = 0, \\
\frac{R}{\gamma - 1} \left( \frac{\partial T}{\partial t} + v \frac{\partial T}{\partial r} \right) + RT \frac{\partial v}{\partial r} = A T^{n_1+1} \rho^{n_2-1} 
\]

subject to laws \( p = \rho v T, \quad \varepsilon = RT(\gamma - 1) \) where \( p \)-pressure, \( \varepsilon \)-specific internal energy. For determination purposes, let us suppose that the piston enters in the gas \((v_0 > 0)\).

In mechanics, similarity and dimensional methods \[4\] are widely used for obtaining self-similarity solutions to the system of Eqs. (1)–(3). With these methods, the problem of finding unknown functions which are characterised by the self-similarity movement and which depend on variables \( r, t \), is reduced to that of finding the function solely dependent on the self-similarity variable \( s \). So in the case under consideration, problem (1)–(3) is reduced to solving the boundary problem of the system of ordinary differential equations with one parameter.

The analysis of the dimensions shows that the solution to problem (1)–(3) will be one of self-similarity, if it satisfies the following conditions:

\[
2n_0 n_1 + (n_0 - 1)(n_1 + n_2 - 1) + 1 = 0, \\
n_0 + 1 > 0, \\
n_0 > -(l + 1)/(l + 3).
\]
In the above case, there exists a unique non-dimensional combination of independent variables \( r \) and \( t \), such as \( s = r/(v_0 t^{n_0+1}) \), and unknown functions may be expressed in the following way:

\[
\begin{align*}
\rho &= \delta(s)\varrho_0 v_0 t^{(n_0+1)/2}, \\
v &= \alpha(s)v_0 t^{n_0}, \\
T &= \varphi(s)\beta_0^{-1} v_0^2 t^{2n_0},
\end{align*}
\]

where \( \delta(s) \), \( \alpha(s) \) and \( \varphi(s) \) are non-dimensional functions of density, speed and temperature satisfying the following system of differential equations with respect to \( s \):

\[
\begin{align*}
[(n_0 + 1)\delta - (n_0 + 1)s - \alpha] \delta + \delta\alpha &= \sigma_0 \delta^{n_1+n_2-1} + m_1, \\
\delta[n_0 \alpha - (n_0 + 1)s - \alpha] + \delta\varphi + \varphi\delta &= 0, \\
\frac{1}{\gamma - 1} \left[ 2n_0 \varphi - ((n_0 + 1)s - \alpha) \varphi \right] + \varphi\alpha &= \sigma_0 \delta^{n_1+n_2-1} \varphi^{n_1},
\end{align*}
\]

where the dot denotes a derivative with respect to \( s \), and the non-dimensional constant \( \sigma_0 \) is expressed as

\[
\sigma_0 = A \delta_0^{n_1+n_2-1} v_0^{-1} \beta_0^{n_1+n_2-1-2n_1}
\]

In the non-dimensional form, condition (3) given on piston can be rewritten in the following way, when \( r = r^*(t) \):

\[
\alpha(s_0) = 1, \quad s_0 = r^*/(v_0 t^{n_0+1}) = 1/(n_0 + 1).
\]

In the case of non-dimensional variables, initial conditions (2) are transformed into those that are given when \( s = \infty \). Nevertheless, as gas-kinetic perturbations are distributed in space at a finite speed, the said conditions must be given at some finite point \( s = s_1 \)

\[
\alpha(s_1) = 0, \quad \varphi(s_1) = 0, \quad \delta(s_1) = s_1^{1/n_1}
\]

If \( s = r_1/(v_0 t^{n_0+1}) \), where \( r_1 > r^* \) is the co-ordinate, characterising the situation of a shock-wave from moving in front of the piston, then when \( r = r_1 \) conditions must be satisfied, relating gas parameters in front and behind the shock-wave front. For this reason, the 'disappearance' of mass from space is not of a superficial but of a volumetric nature, and the conservation laws of shock-wave fronts, expressed as ordinary conditions of
Hugonio, would be the following:
\[
\begin{align*}
q_l(\dot{r}_1 - v_l) &= q_r\dot{r}_1, \\
q_l(\dot{r}_1 - v_l) + p_l &= q_r\dot{r}_1^2, \\
\frac{1}{2}(\dot{r}_1 - v_l)^2 + \frac{\gamma}{\gamma - 1} \cdot \frac{p_l}{q_l} &= \frac{1}{2}\dot{r}_1^2,
\end{align*}
\]
(8)

where the gas parameters behind the shock-wave front are denoted by index \( l \) and those in front of it by index \( r \), and \( \dot{r}_1 \) is the absolute speed of the shock-wave front and \( q_r = q_0 r_1' \). Applying the formulae (4) and (8) and the conditions \( s_1 = r_1/(v_0 t_{n_0}^{-1}) \), we will get:
\[
\begin{align*}
\delta_l &= (\gamma + 1)s_1^l/(\gamma - 1), \\
\alpha_l &= 2(n_0 + 1)s_1/(\gamma + 1), \\
\varphi_l &= \beta_l/\delta_l = 2(\gamma - 1)[(n_0 + 1)^2 s_1/(\gamma + 1)]^2
\end{align*}
\]
(9)

So, when the shock-wave front is situated at point \( s_1 \), the gas parameters behind the front satisfy conditions (7) and those in front of it conditions (9). It is necessary to find solutions \( \alpha(s), \delta(s), \varphi(s) \) to the system of Eq. (5), between the piston and the shock-wave front, and satisfy conditions (6) and (9).

It is convenient to change variables such as:
\[
\tau = (s_1 - s)(s_0 - s_1)^{-1}, \quad u = (n_0 + 1)s - \alpha.
\]

This being done, problems (5), (6) and (9) may be rewritten in the following way:
\[
\begin{align*}
\dot{u} &= -(s_0 - s_1)(u^2 - \gamma \varphi)^{-1} \{2n_0 + (n_0 + 1)(l + \gamma)[\varphi + n_0(n_0 + 1)] \\
&\quad - (s_1 - \tau(s_0 + s_1))u - 2(n_0 + 1)u^2 + \gamma \sigma \delta n_1 - n_0^{-1} - \sigma n_1 - n_0^{-1} \}, \\
\dot{\varphi} &= \varphi(s_0 - s_1)u^{-1}[2n_0 + (n_0 + 1)(\gamma - 1) + (\gamma - 1)\sigma n_1 - n_0^{-1} - \gamma \sigma n_1 - n_0^{-1} - \\
&\quad - (\gamma - 1)\dot{u}(s_0 - s_1)^{-1}], \\
\varphi(0) &= (\gamma - 1)(\gamma + 1)^{-1}(n_0 + 1)s_1, \quad \delta(0) = (\gamma + 1)(\gamma - 1)^{-1}s_1^l, \\
\varphi(1) &= 2(\gamma - 1)(\gamma + 1)^{-2}(n_0 + 1)^2 s_1^2
\end{align*}
\]
(10)

Hence, we obtain the system of ordinary differential equations dependent on parameter \( \lambda = s_1 \).
2. ON ONE ITERATIVE METHOD FOR SOLVING
THE BOUNDARY VALUE PROBLEM OF A FIRST ORDER
ODE WITH ONE PARAMETER ON A MULTIPROCESSOR

2.1. Statement of the Problem

Let us consider the problem of finding a value, such as \( \lambda^* \), of parameter \( \lambda \) and vector function \( X^*_\lambda(t) = (x^1_\lambda(t), \ldots, x^N_\lambda(t)) \), both of which satisfy the system of equations

\[
\frac{dX_\lambda}{dt} = F(t, X, \lambda), \quad t \in (0, 1),
\]

and the following conditions

\[
X_\lambda(0) = \psi(\lambda), \quad x^1_\lambda(1) = 0
\]

where \( \lambda \in (\Lambda_0, \Lambda_1) \), and \( F(t, X_\lambda, \lambda) = \{f^1(t, X_\lambda, \lambda), \ldots, f^N(t, X_\lambda, \lambda)\} \), and \( \psi(\lambda) = \{\psi^1(\lambda), \ldots, \psi^N(\lambda)\} \), are vector functions, \( f^1: [0, 1] \times \mathbb{R}^N \times (\Lambda_0, \Lambda_1) \to \mathbb{R}^1 \), \( \psi^i: (\Lambda_0, \Lambda_1) \to \mathbb{R}^1 \), \( i = 1, \ldots, N \). Vector functions \( F \) and \( \psi \) satisfy conditions shown below.

Together with the problem (13)-(14), let us consider the equivalent of its Cauchy's problem

\[
\frac{dX_\lambda}{dt} = F(t, X_\lambda, \lambda), \quad X_\lambda(0) = \psi(\lambda)
\]

with equation with regard to \( \lambda \)

\[
x^1_\lambda(1) = \psi^1(\lambda) + \int_0^1 f^1(\tau, X_\lambda, \lambda)d\tau = 0
\]

Therefore, problem (13)-(14) is reduced to solving a non-linear equation.

2.2. On Some Iterative Methods for Solving Non-linear Equations
on Parallel Computers

Before we formulate the algorithm for solving non-linear equations, we will define the order of convergence.

Let \( \{\varepsilon_i\}_{i=0}^\infty \) be the sequence that converges to zero, \( \varepsilon_i \neq 0 \), and let \( \lambda \geq 1 \) be a real number.

**Definition 1** If

\[
\lim_{i \to \infty} \frac{|\varepsilon_{i-1}|}{|\varepsilon_i|^\lambda} = q > 0,
\]
then, as regards the sequence \( \{ \varepsilon_i \}_{i=0}^{\infty} \), we shall say that the A-convergence with order \( \lambda \) takes place.

**Definition 2** If

\[
\lim_{i \to \infty} (-\ln |\varepsilon_i|)^{1/i} = \lambda
\]  

(18)

then, as regards the sequence \( \{ \varepsilon_i \}_{i=0}^{\infty} \), we shall say that the B-convergence with order \( \lambda \) takes place.

**Definition 3** If

\[
|\varepsilon_i| \leq \gamma \theta^i, \quad \text{when } i \geq i_0 \quad \text{and } \lambda > 1,
\]  

(19)

where \( \gamma > 0, \) \( 0 < \theta < 1 \) are constants, then as regards the sequence \( \{ \varepsilon_i \}_{i=0}^{\infty} \), we will say that the C-convergence with order \( \lambda \) takes place.

We can show that from A-convergence follows B-convergence and C-convergence. The inverse rule in general case is not true. Likewise, we can show that from (18) does not follow (19) and *vice versa*.

Let us devise the iterative methods for solving non-linear equations. Let \( f(x) \) be the function of one real variable and let us assume that we can find interval \( I = (a, b) \) so that the equation \( f(x) = 0 \) has a unique root \( \bar{x} \) and \( f(x) \in C^k(I) \). We demand that \( f'(x) \neq 0 \), when \( x \in I \). That is why \( f(\bar{x}) = 0 \), and we can find the neighbourhood of zero \( I^* \), where \( g \equiv f^{-1} \) and \( g \in C^k(I^*) \).

Let us now consider the iterative sequence of N-dimensional vectors

\[
\vec{X}^{(i)} = (x_1^{(i)}, x_2^{(i)}, \ldots, x_N^{(i)}), \quad i = 0, 1, 2, \ldots
\]  

(20)

each vector being an approximate value of \( \bar{x} \).

Then, we can show that, when \( i \to \infty \) and conditions are defined, vector \( \vec{X}^{(i)} \) converges to \( \bar{x} \).

Let \( \vec{X}^{(0)} = (x_1^{(0)}, \ldots, x_{N-1}^{(0)}, x_N^{(0)}) \), a initial approximation to \( \bar{x} \) from \( I \), and the values of function \( f(x) \) and its derivatives at the these points are given:

\[
\begin{align*}
   f(x_1^{(0)}) &= y_1^{(0)}, & f(x_N^{(0)}) &= y_N^{(0)}, \\
   f'(x_1^{(0)}), & \ldots, & f'(x_N^{(0)}), \\
   \vdots \\
   f^{(\alpha)}(x_1^{(0)}), & \ldots, & f^{(\alpha)}(x_N^{(0)}), \\
   (\alpha_j \leq k + 1, \quad j = 1, \ldots, N).
\end{align*}
\]  

(21)

If the values of the derivatives of function \( y = f(x) \) are known, then it is not difficult to find the derivatives of the inverse function [5].
Evidently, \( x = g(0) \). For finding the approximations to \( x \), we replace the function \( g(y) \) by Hermite's interpolating polynomial of a subset of the set \( \{y^{(i)}_j\}_{i=1}^N \), and then compute the polynomial's value at point 0.

Let \( M \) denote the number of knots used in Hermite's interpolating polynomial, \( 2 \leq M \leq N - 1 \). For the \( j \)-component of vector \( X^{(i)} \), interpolating polynomial of Hermite \( H_j^{(i)}(y) \) is constructed. We will choose the knots for interpolating polynomials in the following way: Firstly, let us consider the set of indexes \( A = \{1, 2, \ldots, N\} \) and choose \( N \) distinct subset \( A_j \) of the said set. The number of elements in each subset is equal to \( M \), and let \( j \in A_j \). For convenience of realization, it is necessary that all the Hermite's polynomials have the same order \( m - 1 \), where \( m \leq k \).

For the knots of polynomial \( H_j^{(i)}(y) \), we will take points \( \{y^{(i)}_s\}, s \in A_j \). It is evident that the knots of interpolation must satisfy the following conditions:

\[
\begin{align*}
H_j^{(i)}(y^{(i)}_s) &= x^{(i)}_s, \\
[H_j^{(i)}(y^{(i)}_s)]' &= [x^{(i)}_s]', \quad s \in A_j \\
&\quad \cdots \cdots \\
[H_j^{(i)}(y^{(i)}_s)]^{(\alpha_k-1)} &= [x^{(i)}_s]^{(\alpha_k-1)},
\end{align*}
\]  

(22)

where \( \sum_{s \in A_j} \alpha_{js} = m \), and \( \alpha_{js} \leq \alpha_{sj}, j = 1, \ldots, N \).

Let us consider the following iterative process, in which the \( j \)-component of vector \( \tilde{X}^{(i+1)} \) is computed in the following way:

\[
x^{(i+1)}_j = H_j^{(i)}(0), \quad j = 1, \ldots, N.
\]  

(23)

Hence, we will obtain the sequence of vectors \( \{\tilde{X}^{(i)}\}_{i=0}^{\infty} \).

It should be noted that this algorithm should be solved on \( N \) parallel processors, algorithm is suitable for implementation on a multiprocessor of type SIMD.

If \( \alpha_1 = \alpha_2 = \cdots = \alpha_N = 1 \), the method used is the same as that used for the iterative algorithm in [6].

Let us prove the convergence of method (20), (23).

To this end, a matrix interpretation of this method is used, analogical interpretation was used in [7], where well-known iterative methods are employed and their speed of \( B \)-convergence is estimated. Below, an analogical interpretation is applied to iterative method (20), (23), and the speed of its \( C \)-convergence is estimated.

For constructing polynomial \( H_j^{(i)}(y) \), let us assume that the values of the function and its derivatives \( \{y^{(i)}_s\}, \{g(y^{(i)}_s)\}, \{g^{(\alpha_k-1)}(y^{(i)}_s)\}, s \in A_j \), are used.
Let us consider vector
\[ \lambda_j = (t_{j1}, t_{j2}, \ldots, t_{jN}) \]  \hspace{1cm} (24)
where \( t_{js} = 0 \) if \( s \not\in A_j \) and \( t_{js} = \alpha_{js} \) if \( s \in A_j \).

So, to each polynomial \( H_j^{(i)}(y) \) corresponds vector \( \lambda_j, j = 1, 2, \ldots, N \), and matrix \( \Lambda \) with lines \( \lambda_j \) corresponds entirely to the iterative method.

Let \( \hat{X}^{(0)} = (x_1^{(0)}, \ldots, x_N^{(0)}) \) be the initial approximation of iterative process (20), (23), function \( f^{(m)}(X) \) being limited at \( I \) and function \( g^{(k)}(y) \neq 0 \) at \( I^* \).

Denote: \( q = \max_{1 \leq j \leq N} |\bar{x} - x_j^{(0)}| = ||\varepsilon^{(0)}||, \varepsilon^{(0)} = (\varepsilon_1^{(0)}, \varepsilon_2^{(0)}, \ldots, \varepsilon_N^{(0)}) \),
\[ p = \sup_{y \in I^*} \frac{|g^{(m)}(y)|}{m!} \sup_{x \in I} |f'(x)|^m. \]  \hspace{1cm} (25)

**Theorem 1**  \hspace{1cm} Let initial values be chosen from neighbourhood \( I^0 \subseteq I \), so that the points at \( I^0 \) satisfy inequality \( pq^{n-1} < 1 \). Let it be assumed that the set \( A_j \) contains points \( j \) and \( j + 1, j = 1, 2, \ldots, N \). Then, for the iterative process (20), (23) it is valid the C-convergence with speed of convergence equal to the spectral radius of matrix \( \Lambda - m \).

**Proof**  \hspace{1cm} If \( y_j^{(i)} = 0, j = 1, \ldots, N \) then \( x_j^{(i)} = \bar{x}, j = 1, \ldots, N \) and, therefore, the theorem does not need any proof.

Let us consider the case when \( y_j^{(i)} \neq 0, j = 1, \ldots, N \). Let us also suppose that the knots of interpolation are not multiple, i.e., \( y_j^{(i)} \neq y_k^{(i)} \), when \( j \neq k \).

If \( x_j^{(i)} = x_k^{(i)} \), when \( j \neq k \), then \( x_j^{(i)} \) may be replaced by a value \( x \) from interval \( (\min_j x_j^{(i)}, \max_j x_j^{(i)}) \). If all \( x_j^{(i)} \) are equal for certain \( i \), then it will be impossible to continue the iterative method, so we must choose other initial approximations from interval \( I^* \).

For this reason \( \bar{x} = g(0) \approx H_j^{(i)}(0) = x_j^{(i+1)}, j = 1, \ldots, N \), from the remainder of the term of Hermite’s interpolating formula [8] we can write
\[ \varepsilon_j^{(i+1)} = \left| \frac{g^{(m)}(\eta_j)}{m!} \prod_{s \in A_j} (-y_j^{(i)})^{\alpha_s} \right| = \left| \frac{g^{(m)}(\eta_j)}{m!} \prod_{s \in A_j} \frac{\varepsilon_s^{(i)}}{g'\left(\xi_s\right)} \right| \]  \hspace{1cm} (26)
where \( \varepsilon^{(k)} = |\bar{x} - x^{(k)}|, \eta_j \in I^*, \xi_s \in \left( \min(0,y_s^{(i)}), \max(0,y_s^{(i)}) \right), s \in A_j, j = 1, \ldots, N \).

Let us define \( N \) majorant sequences of positive numbers \( \varepsilon_j^{(i)} (j = 1, \ldots, N) \) in the following way:
\[ \varepsilon_j^{(0)} = q, \varepsilon_j^{(i+1)} = \prod_{s \in A_j} (\varepsilon_j^{(i)})^{\alpha_s}, j = 1, \ldots, N, \; i = 1, 2, \ldots \]  \hspace{1cm} (27)
Then, the inequality \( e_j^{(t)} \geq \epsilon_j^{(t)} \) is necessary and easy to prove by mathematical induction.

Consequently, we consider the case \( y_j^{(t)} \neq 0 \), then \( \epsilon_j^{(t)} \neq 0 \) \( (j = 1, \ldots, N) \). Apart from this, \( g^{(k)}(x) \neq 0 \) when \( x \in I^* \) because \( \epsilon_j^{(i+1)} \neq 0 \). From this we can conclude that \( \epsilon_j^{(t)} \neq 0 \) \( (j = 1, \ldots, N, i = 0, 1, 2, \ldots) \). Taking this into account, if we take the logarithm from both sides of equalities (27) we can write, \( i = 1, 2, \ldots \)

\[
 b_j^{(i+1)} = 1 + \sum_{s \in A_j} \alpha_{is} b_s^{(i)} \quad j = 1, \ldots, N, \tag{28}
\]

where \( b_j^{(i)} = \log_e e_j^{(i)}, s = 1, \ldots, N \).

Let us consider the homogenous system corresponding to system (28):

\[
 \tilde{a}^{(i+1)} = \Lambda \tilde{a}^{(i)}, \tag{29}
\]

where \( \tilde{a}^{(i)} = (a_1^{(i)}, \ldots, a_N^{(i)}) \), and where matrix \( \Lambda \) is defined above by equalities (24). For the solution of the system of difference Eq. (29), we will use as usual the form \( \tilde{a}^{(i)} = \lambda^i \tilde{z} \). Then, from (29) we will get

\[
 \lambda^i (\Lambda - \lambda \tilde{E}) \tilde{z} = 0 \quad \text{or} \quad (\Lambda - \lambda \tilde{E}) \tilde{z} = 0. \tag{30}
\]

Therefore, to obtain a non-trivial solution to difference system (29), \( \lambda \) must be the eigenvalue of matrix \( \Lambda \) and \( \tilde{z} \) the corresponding eigenvector.

As all the elements of matrix \( \Lambda \) are non-negative, the said matrix has a positive eigenvalue \( \lambda \), which is equal to its spectral radius \( \varrho(\Lambda) \), and a right eigenvector with positive components corresponding to the said eigenvalue exists [9].

Let us now consider the matrix \( \Lambda' = (1/m) \Lambda \). Evidently, the matrix \( \Lambda' \) is stochastic and its spectral radius is equal to 1 : \( \varrho(\Lambda') = 1 \) [9]. As a result of this \( \varrho(\Lambda) = m \). Without loss of generality, we believe that \( \lambda_1(\Lambda) = m \).

It is easy to verify the non-reducibility of matrix \( \Lambda \) because its directed graph is very connected. Likewise, it is easy to show that all elements of matrix \( \Lambda^{N-1} \) are positive, which demonstrates that the matrix's index of imprimiting is equal to 1. Hence, matrix \( \Lambda \) is primitive and, therefore, has a dominating positive eigenvalue \( m \) with multiple 1 [9].

Consequently, the general solution to the difference system (29) can be written in the following way:

\[
 \tilde{a}^{(i)} = \alpha_1 m^i \tilde{z}^{(i)} + \sum_{k=2}^{N} \lambda_k^{(i)} \tilde{z}^{(k)} \cdot \sum_{p=1}^{kp} d_{kp} \cdot i^{p-1}, \tag{31}
\]
where \( k_p \) is the multiplier of the eigenvalue \( \lambda_k \), \( N' \) the number of distinct eigenvalues: \( \sum_{k=2}^{N'} k_p = N - 1 \), and \( \bar{z}^{(k)} \) the eigenvector corresponding to eigenvalue \( \lambda_k \), and \( d_1, d_{k_p} \ (p = 1, \ldots, k_p, k = 2, \ldots, N') \) are constants.

The general solution to the non-homogeneous system (28) can be written in the following way:

\[
\bar{z}^{(i)} = b_0 + d_1 m^i \bar{z}^{(1)} + \sum_{k=2}^{N'} \lambda_k^i \bar{z}^{(k)} \sum_{p=1}^{k_p} d_{kp} i^{p-1},
\]

where \( \bar{b}_0 \) is the particular solution to system (28). By immediate verification we can say that for \( \bar{b}_0 \) we can take the following vector \( \bar{b}_0 = (-1/m-1, \ldots, -1/m-1) \). To define constants \( d_1, d_{k_p} \ (p = 1, \ldots, k_p, k = 2, \ldots, N') \), we use the initial values of vector \( \bar{b}^{(i)} \), when \( i = 1 \):

\[
\bar{b}^{(1)} = (b_1^{(1)}, b_2^{(1)}, \ldots, b_N^{(1)}),
\]

where \( b_s^{(1)} = \log e_s^{(1)} = 1 + m \log q, (s = 1, \ldots, N) \) \( (33) \)

As matrix \( (1/m)A \) is non-negative stochastic, then to maximal eigenvalue corresponds right eigenvector \((1,1,\ldots,1)\) and from (32) the linear system of algebraic equations with coefficients \( d_1, d_{k_p} \ (p = 1, \ldots, k_p, k = 2, \ldots, N') \) is obtained as below:

\[
d_{lm} + \sum_{k=2}^{N'} \lambda_k z_s^{(k)} \sum_{p=1}^{k_p} d_{kp} = 1 + m \log q + \frac{1}{m-1}, \quad s = 1, \ldots, N. \quad (34)
\]

From system (34) it is easy to define value \( d_1 \):

\[
d_1 = \frac{1}{m-1} + \log q. \quad (35)
\]

Expression (32) can be rewritten as \( \bar{z}^{(i)} = m^i (d_1 \bar{z}_1 + D^{(i)}) \), where

\[
D^{(i)} = \frac{1}{m^i} \bar{b}_0 + \sum_{k=2}^{N'} \lambda_k \sum_{p=1}^{k_p} d_{kp} i^{p-1} \to 0, \quad \text{when } i \to \infty.
\]

Using the above expression, we can write the following expression for values \( e_s^{(i)} \):

\[
e_s^{(i)} = p^{m^i (d_1 - D^{(i)}),} \quad s = 1, \ldots, N. \quad (36)
\]
Let $p \geq 1$. Using equality (35) and the condition of theorem $pq^{n-1} < 1$, we can write

$$d_i = \log \left( \frac{1}{p^{n-1}q} \right) = \log(pq^{n-1})^{\frac{1}{m-1}} < 0.$$  

If $p < 1$ we can analogically show that $d_i > 0$.

That is why $d_i + D_s \to d_i$, when $i \to \infty$, and we can then choose a natural number $i_0$ so that, when $i \geq i_0$, the following estimation is true:

$$e^{(i)} \leq \delta_m^i = \left[ \left( \frac{1}{p^{n-1}q} \right)^{1/2} \right]^m.$$  

Finally, the following estimation is obtained:

$$e^{(i)} \leq \theta^m, \quad \text{where} \quad \theta = \left( \frac{1}{p^{n-1}q} \right)^{1/2}, \quad j = 1, \ldots, N, \quad i \geq i_0.$$  

The theorem is totally proved.

If we apply the method used in [6], together with the remainder of the term of Hermite’s interpolating formula, we can also prove the $B$-convergence of this iterative process.

**Theorem 2**  
Assume that the conditions of Theorem 1 are fulfilled. Then, for iterative method (20), (23) the $B$-convergence with order $m$ takes place.

From Theorems 1 and 2 it follows that the convergence of iterative process (20), (23) depends on the choice of initial elements. Local convergence takes place in the conditions of these theorems. However, from [10] we can construct an iterative procedure that allows us to find the necessary initial values from $I^*$ in a finite number of steps. After obtaining these initial values, the iterative process (20), (23) is accomplished automatically.

The domain of convergence of iterative process (20), (23), with iterative finding of initial approximation, is the whole interval.

Let us estimate the expression $q = \max_{1 \leq j \leq N} |\tilde{y} - x_j^{(0)}| = \max_{1 \leq j \leq N} |g(0) - g(x_j^{(0)})|$:

$$q \leq \max_{1 \leq j \leq N} \frac{|f(x_j^{(0)})|}{\|f'(x)\|}.$$  

If initial values for iterative process (20), (23) are chosen from neighbourhood $I^0$, whose points satisfy the inequality

$$p^{\frac{1}{m-1}} \cdot \max_{1 \leq j \leq N} \frac{|f(x_j^{(0)})|}{\|f'(x)\|} < 1 \quad (37)$$  


then the conditions of Theorem 1 will be satisfied and, therefore, the iterative method (20), (23) will be convergent.

For the $j$-processor $j = 1, 2, \ldots, N$ we consider equation $f(x) = \alpha^{(0)} f(x^{(0)}_j)$, instead of equation $f(x) = 0$, where $x^{(0)}_j \in I$ at any point from the interval $(a, b)$ and $\alpha^{(0)}$, $i = 0, 1, 2, \ldots$ is the sequence of numbers from $[0, 1]$. The choice of this sequence is considered below. For solving this equation on a $j$-processor, we use formula (23), but we must also take into account that the values of Hermite's polynomial are computed for the function $f(x) - \alpha^{(0)} f(x^{(0)}_j)$, and constants $\alpha^{(0)}$ are determined in the following way:

$$
\alpha^{(i)} = \max \left[ 0, 1 - (i + 2)/\left( 4p^{i+1} \cdot \max_{1 \leq j \leq N} |f(x^{(0)}_j)|/\|f'(x)\| \right) \right],
$$

$$
\text{for } i = 0, 1, \ldots
$$

Therefore, we also obtain the sequence of $N$-dimensional vectors $X^{(i)}$, $i = 0, 1, \ldots$ We will call this iterative process the modified iterative method.

Denote by $\phi^{(i)} = \max_{1 \leq j \leq N} |f(x^{(i)}_j)| - \alpha^{(i)} f(x^{(0)}_j)$.

Lemma (a) Sequence $\alpha^{(i)}$ ($i = 0, 1, \ldots$) is non-increasing, i.e., $\alpha^{(i+1)} \leq \alpha^{(i)}$, where $\alpha^{(i)} \in [0, 1]$, $i = 0, 1, \ldots$; (b) If $f'(x) \neq 0$, then for each $x^{(0)}_j$, $j = 1, \ldots, N$, $i = 0, 1, 2, \ldots$, obtained by the modified iterative method, where $x^{(0)}_j \in I$ ($j = 1, \ldots, N$), the inequality

$$
2p^{i+1} \phi^{(i)}/\|f'(x)\| \leq 1
$$

is satisfied.

Proof Firstly, we will prove the validity of sentence (a), using mathematical induction. Let $i = 0$, then

$$
\alpha^{(0)} = \max \left[ 0, 1 - \|f'(x)\|/\left( 2p^{i+1} \cdot \max_{1 \leq j \leq N} |f(x^{(0)}_j)| \right) \right] \in [0, 1].
$$

is obtained from (38)

From this reason, function $(i + 2)\|f'(x)\|/(4p^{i+1} \max_{1 \leq j \leq N} |f(x^{(0)}_j)|)$ increases towards $i$ and, therefore, it is just inequality $\alpha^{(i)} \geq \alpha^{(i+1)}$, $i = 0, 1, 2, \ldots$. Consequently, sentence (a) is proved.
The second part of the lemma is also proved by mathematical induction. Let \( i = 0 \), then

\[
2p^{m-1} \phi^0 / \| f'(x) \| = 2p^{m-1} \| f'(x) \|^{-1} \max_{1 \leq j \leq N} |f(x_j^0)| \cdot \min \left[ 1, \left( 2p^{m-1} \| f'(x) \|^{-1} \max_{1 \leq j \leq N} |f(x_j^0)| \right)^{-1} \right] \leq 1.
\]

Let us assume that inequality (39) satisfies \( i \), and prove the validity of this inequality for \( i + 1 \).

Next, we will suppose that equation \( f(x) = \alpha^{(i)}f(x_j^0), \ j = 1, 2, \ldots, N, \ 0 \leq \alpha^{(i)} \leq 1 \), has a unique root for arbitrary \( i = 0, 1, 2, \ldots \) at interval \( I \). Nevertheless, for proving the above-mentioned lemma, let us suppose that the roots are \( \bar{x}_j \ (j = 1, \ldots, N, \ i = 0, 1, 2, \ldots) \).

Let us consider

\[
\phi^{(i+1)} \leq \| f'(x) \| \max_{1 \leq j \leq N} |x_j^{(i+1)} - \bar{x}_j + (\alpha^{(i)} - \alpha^{(i+1)}) \max_{1 \leq j \leq N} |f(x_j^0)| \| f(x_j^0) \|, \quad (40)
\]

Let us estimate the following expression: \( x_j^{(i+1)} - \bar{x}_j \)

\[
| x_j^{(i+1)} - \bar{x}_j | \leq \sup_{y \in I} |g^{(m)}(y)|/m! \cdot \prod_{j \in A_y} |f(x_j^{(i)}) - \alpha^{(i)}f(x_j^0)|^{\alpha_{ij}} \leq \sup_{y \in I} |g^{(m)}(y)|/m! \cdot (2p^{m-1})^{-m} \cdot \| f'(x) \|^m \leq 2^{-m} p^{-m}.
\]

Substituting the above estimation we have:

\[
\phi^{(i+1)} \leq \| f'(x) \| \cdot 2^{-m} p^{-m} \cdot \max_{1 \leq j \leq N} |f(x_j^0)|.
\]

If \( \alpha^{(i)}, \alpha^{(i+1)} \neq 0 \), then

\[
\phi^{(i+1)} \leq (2p^{m-1})^{-1} \cdot \| f'(x) \|.
\]

From this we can write

\[
2p^{m-1} \| f'(x) \|^{-1} \phi^{(i+1)} \leq 1.
\]
Let \( \alpha^{(i)} \neq 0 \), \( \alpha^{(i+1)} = 0 \), then

\[
\max |f(x^{(i-1)}_j)| \leq (4p^{(i-1)})^{-1}\|f'(x)\| + \alpha^{(i)} \max_{1 \leq j \leq N} |f(x^{(0)}_j)| =
\]

\[
= \left[ 4p^{(i-1)}\|f'(x)\|^{-1} \cdot \max_{1 \leq j \leq N} |f(x^{(0)}_j)| - (i + 1) \right] \cdot (4p^{(i-1)})^{-1}\|f'(x)\| \leq
\]

\[
\leq [(1 + 3) - (i + 1)] \cdot (4p^{(i-1)})^{-1}\|f'(x)\| = (2p^{(i-1)})^{-1}\|f'(x)\|
\]

and we obtain the required inequality again.

In this way, the lemma is totally proved.

**Theorem 3** Let equation \( f(x) = 0 \) has a unique root at interval \( I \), \( f(x) \in C^k(I) \) and \( f'(x) \neq 0 \), if \( x \in I \). Then, for arbitrary \( x^{(0)}_j \in I (j = 1, \ldots, N) \) after a finite number \( i_0 \) of steps, the modified iterative method gives points \( x^{(i_0)}_j \) \( (j = 1, \ldots, N) \), for which \( \alpha^{(i)} = 0 \), when \( i \geq i_0 \), and the condition of convergence (37) of iterative process (20), (23) is satisfied.

**Proof** If we take into account expression (38) for parameters \( \alpha^{(i)} \), \( i = 1, 2, \ldots \), then we can conclude that for arbitrary \( x^{(0)}_j \in I (j = 1, \ldots, N) \) we can always find a natural number \( i_0 \), so that \( \alpha^{(i_0)} = 0 \) and all \( \alpha^{(i)} = 0 \), when \( i \geq i_0 \). Taking into account the lemma, we can conclude that inequality (39) is satisfied, when \( i = i_0 \). This means that, if we take points \( x^{(i_0)}_j \) \( (j = 1, \ldots, N) \) for initial approximations, condition (37) of convergence of iterative method (20), (23) will be satisfied.

### 2.3. The Algorithm For Solving Problem (15) – (16)

For solving problem (15) – (16) we will consider the iterative method that uses the methodology explained in (2.2). We will apply interpolating formulae that only use two knots. So, to perform the algorithm on a multi-processor, it is necessary to use three processors. It should be noted that in the case of the other interpolating formulae and, hence, the use of many processors, we can consider the applied results from (2.2) analogically.

Let Cauchy’s problem (15) for each \( \lambda, \lambda \in (\Lambda_0, \Lambda_1) \) has solution \( X_\lambda(t) = (\hat{x}_\lambda^{(1)}(t), \ldots, \hat{x}_\lambda^{(N)}(t)) \), where \( \hat{x}_\lambda^{(1)}(t) \in C^1[0, 1] \).

For constructing the iterative process, the following conditions must be fulfilled:

\[
\frac{d}{d\lambda} \hat{x}_\lambda^{(1)}(1) \neq 0,
\]

where \( \hat{x}_\lambda^{(1)} : (\Lambda_0, \Lambda_1) \rightarrow (\tilde{\Lambda}_0, \tilde{\Lambda}_1) \). Then, function \( \varphi \equiv [\hat{x}_\lambda^{(1)}(1)]^{-1} \) exists on \( (\Lambda_0, \Lambda_1) \). Evidently, function \( \varphi \) has the same smoothness as \( \hat{x}_\lambda^{(1)}(1) \).
Using initial values \( \lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)} \), we can then construct the following sequence of solutions to problem (15)–(16):

\[
\frac{dX_{\lambda_i}}{dt} = F(t, X_{\lambda_i}, \lambda_i^{(j)}), \quad X_{\lambda_i} = \psi(\lambda_i^{(j)})
\]

(42)

\[
\lambda_i^{(i+1)} = H_j^{(i)}(0),
\]

(43)

where \( i = 0, 1, 2, \ldots \) is the numeration of iterations, \( j = 1, 2, 3 \) the numeration of processors and \( H_j^{(i)}(z) \), Lagrange’s inverse interpolating polynomial in case (a) or Hermite’s inverse interpolating polynomial in case (b), which are constructed for knots \( z_j^{(i)} = \bar{x}_{\lambda_i}^{(j)}(1) \), \( z_j^{(i)} = \bar{x}_{\lambda_i}^{(j)}(1) \), \( j = 1, 2, 3 \), \( z_4^{(i)} \equiv z_1^{(i)} \).

The whole process can be graphically expressed in the following way:

So as to simplify the transmission, in Hermite’s polynomial only first order derivatives with respect to \( \lambda \) are used. It should be noted that, in this case, we must require more smoothness of vector-function \( f \) regarding its arguments. The values of first order derivatives with regard to \( \lambda \) of function \( \overline{x}_\lambda(1) \) can be computed by the following Cauchy’s problem:

\[
\frac{dV_{\lambda_i}}{dt} = \sum_{k=1}^{N} \frac{\partial f^k(t, X_{\lambda_i}^{(j)}, \lambda_i^{(j)})}{\partial (X_{\lambda_i}^{(j)})} u_{\lambda_i}^k(t) + \frac{\partial f(t, X_{\lambda_i}^{(j)}, \lambda_i^{(j)})}{\partial \lambda},
\]

(44)
\begin{equation}
  u_{\lambda_i}(0) = \frac{d}{d\lambda} \psi(\lambda_i^{(i)}),
\end{equation}

where $u_{\lambda}(t) = (u_1^{(i)}(t), \ldots, u_N^{(i)}(t))$ denotes $\frac{\partial \psi_i(t)}{\partial \lambda}$.

For proving the convergence of iterative process (42)-(43) in case (a), the following theorem can be used.

**Theorem 4** Assume that $f(t, X, \lambda) \in C^2$ $(\Lambda_0, \Lambda_1)$ for every fixed $t$ and $X$, $\psi(\lambda) \in C^2$ $(\Lambda_0, \Lambda_1)$ and constants $L_1$ and $L_2$ exist so that

\begin{equation}
  \|F(t, X, \lambda_1) - F(t, Y, \lambda_2)\| \leq L_1 \|X - Y\| + L_2 |\lambda_1 - \lambda_2|.
\end{equation}

Let condition (41) be fulfilled.

Denote by $p = ||\varphi^{(2)}(\xi)|| \cdot ||u_1^{(1)}(1)||^2 \cdot 2^{-1}; \eta = \max_{1 \leq j \leq 3} |\lambda_j^{(0)} - \lambda^*|$. If

\begin{equation}
  p\eta < 1,
\end{equation}

then $\lambda_j^{(i)} \to \lambda^*$, when $i \to \infty$, $j = 1, 2, 3$; also,

\begin{equation}
  |\lambda_j^{(i)} - \lambda^*| \leq p^{-1}(\eta p)^2.
\end{equation}

and

\begin{equation}
  \|\bar{X}_{\lambda_i^{(i)}}(t) - \bar{X}_{\lambda^*}(t)\| = O[(p\eta)^2].
\end{equation}

**Proof** From (15) it follows that

\begin{equation}
  \bar{X}_{\lambda_1}(t) - \bar{X}_{\lambda_2}(t) = \psi(\lambda_1) - \psi(\lambda_2) + \int_0^t [F(\tau, \bar{X}_1, \lambda_1) - F(\tau, \bar{X}, \lambda_2)]d\tau.
\end{equation}

If we apply (46), we get

\begin{align*}
  \|\bar{X}_{\lambda_1}(t) - \bar{X}_{\lambda_2}(t)\| &\leq M \cdot |\lambda_1 - \lambda_2| \\
  &+ \int_0^t [L_1 (|\bar{X}_{\lambda_1}(\tau) - \bar{X}_{\lambda_2}(\tau)| + L_2 |\lambda_1 - \lambda_2|) d\tau \\
  &= (M + L_2) \cdot |\lambda_1 - \lambda_2| \\
  &+ L_1 \int_0^t \|\bar{X}_{\lambda_1}(\tau) - \bar{X}_{\lambda_2}(\tau)| d\tau,
\end{align*}

for every $\lambda_1, \lambda_2 \in (\Lambda_0, \Lambda_1)$, where $M = \max_{1 \leq i \leq N} |\psi^{(i)}(\xi)|$. 

Applying Lemma 3 from [11] with regard to the last inequality, a constant $L > 0$ exists so that

$$
\| \bar{X}_{\lambda_1}(t) - \bar{X}_{\lambda_2}(t) \| \leq L |\lambda_1 - \lambda_2|.
$$

From the last inequality it follows that

$$
\| \bar{X}_{\lambda_{j0}}(t) - \bar{X}_{\lambda^*}(t) \| \leq L |\lambda_{j0} - \lambda^*|.
$$

(50)

Let us now estimate $|\lambda_{j0} - \lambda^*|$. Applying the formula for the remainder of the term of Hermite's interpolating polynomial, we get

$$
|\varepsilon_j^{(i+1)}| = |\lambda_{j0}^{(i+1)} - \lambda^*| = \frac{\varphi^{(2)}(\xi_j) \prod_{s=j}^{i+1} |\lambda_{j0}^{(s)} - \lambda^*|}{2 \varphi'(\xi_j)}.
$$

where, $\xi_j \in (\bar{\lambda}_0, \bar{\lambda}_1), \xi_s \in (\min(0, \bar{x}_{\lambda_{j0}}(1)), \max(0, \bar{x}_{\lambda}^1(1))$, $s = j, j + 1, j = 1, 2, 3.$

Let us now define the following three majorant sequences of positive numbers $\varepsilon_j^{(i)}$:

$$
\varepsilon_j^{(0)} = \eta, \varepsilon_j^{(i+1)} = p \prod_{s=j}^{i+1} \varepsilon_j^{(i)}, \quad j = 1, 2, 3, \quad i = 0, 1, \ldots.
$$

Inequality $\varepsilon_j^{(i)} \geq \varepsilon_j^{(i)}$ is fulfilled and is easily proved by mathematical induction. Then, $\varepsilon_j^{(1)} = p\eta^2 = p^{-1}(p\eta)^2; \varepsilon_j^{(2)} = p^{-1}(p\eta)^2; \ldots, \varepsilon_j^{(i)} = p^{-1}(p\eta)^2$.

Because of (47), $\varepsilon_j^{(i)}$ and consequently $\varepsilon_j^{(i)}$ tend to zero, when $i \to \infty$. So,

$$
|\lambda_{j0}^{(i)} - \lambda^*| < p^{-1}(p\eta)^2i,
$$

and inequality (50) gives estimation (49).

The Theorem 4 is proved.

Let us now consider iterative process (42)–(45) for case (b). Let us assume that Cauchy's problem (44)–(45) for every $\lambda \in (\Lambda_0, \Lambda_1)$ has solution $u_{\lambda}(t), t \in [0, 1]$.

For constructing Hermite's inverse polynomial in (43), we use the values of inverse function $\varphi$ and its derivatives at points $z_j^{(i)} = \bar{x}_{\lambda_{j0}}^1(1), z_{j+1}^{(i)} = \bar{x}_{\lambda_{j+10}}^1(1), j = 1, 2, 3, z_i^{(i)} = z_i^{(i)}$.

For the convergence of iterative process (42)–(45), the following theorem is proved:
THEOREM 5  Assume that $F(t, X, \lambda) \in C^4 (\Lambda_0, \Lambda_1)$ for every fixed $t$ and $X$, $\psi(\lambda) \in C^4(\Lambda_0, \Lambda_1)$ and constants $L_1$ and $L_2$ exist so that

$$\|F(t, X, \lambda_1) - F(t, Y, \lambda_2)\| \leq L_1\|X - Y\| + L_2|\lambda_1 - \lambda_2|.$$ 

Let condition (41) be satisfied.

Denote by $p = \|\psi^{(4)}(\xi)\|\|u^{(1)}_\lambda(1)\|^4 / 24; \eta = \max_{1 \leq j \leq 3} |\lambda_j^{(0)} - \lambda^*|$. If $p\eta^3 < 1$, then $\lambda_j^{(i)} \to \lambda^*$, when $i \to \infty$; $j = 1, 2, 3$; also

$$|\lambda_j^{(i)} - \lambda^*| \leq p^{-1}(\eta^3 p)^4$$

and

$$\|X^0_{\lambda^*}(t) - X^0_{\lambda^*}(t)\| = O((p\eta^3)^4).$$

Theorem 5 is proved in the same way as Theorem 4. It should be noted that the convergence of the method used in this paper can also be proved if some numerical method is applied for solving Cauchy’s problem.

3. NUMERICAL RESULTS

For solving the first order system of ODE with a parameter (10) with initial and boundary conditions (11) and (12) respectively, iterative method (42)-(43), case (a), is applied.

Initial approximations $\lambda_1^{(0)}, \lambda_2^{(0)}, \lambda_3^{(0)}$ are chosen, and Cauchy’s problem is solved on each processor using Euler’s method. Next, on each processor Eq. (12) is solved applying the following formula:

$$\lambda_j^{(i+1)} = \lambda_j^{(i)} = \frac{\lambda_j^{(i)} - \lambda_j^{(i+1)}}{\lambda_j^{(i)}(1) - \lambda_j^{(i+1)}(1)}, \quad j = 1, 2, 3; \quad (51)$$

Next, new approximations $\lambda_1^{(1)}, \lambda_2^{(1)}, \lambda_3^{(1)}$ are obtained, and so on.

Computation carried out for several values of $\gamma, l, n_0, n_1, n_2$ (see Tab. 1).

<table>
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<tr>
<th>$N$</th>
<th>$\gamma$</th>
<th>$l$</th>
<th>$n_0$</th>
<th>$n_1$</th>
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</table>
FIGURE 1 Graphic representation of the solution to system 5 for values $N = 1$. 

$N_0 = -0.1$, $n_1 = 2$, $n_2 = -5/3$, $l = 1$. 

FIGURE 2 Graphic representation of the solution to system 5 for values $N = 2$. 

$n_0 = -0.1$, $n_1 = 5$, $n_2 = 0$, $l = 0$. 
FIGURE 3 Graphic representation of the solution to system 5 for values $N = 3$.

$\gamma = 5/3, I = 1, n_i = 0.5, n_1 = -1, s_2 = 2$. 

FIGURE 4 Graphic representation of the solution to system 5 for values $N = 4$.

$\gamma = 5/3, I = 1, n_i = 0.5, s_i = 1, n_1 = -4/3$. 

In variants 3 - 5, condition \( l = L \) is satisfied, where
\[
L = 2n_0/((n_0 + 1)(\gamma - 1))
\]
and entropy is constant between the piston and the shock-wave front [12]. Graphs 1 - 5 for variants 1 - 5 show the dependence of the functions of density \( \delta(s_1) \), speed \( u(s_1) \) and temperature \( f(s_1) \) behind the shock-wave front on the capacity of resources (channels). Dependence of the situation of the front \( s_1 \) on \( \sigma_0 \) is shown as a discontinuous line. From these graphs it follows that the increase of the influence of the mass source on the movement of gas in front of the piston (i.e., an increase of \( \sigma_0 \)) leads to a decrease in the speed and density.

Apart from using formula (51), system (12) was solved by Newton's method and Chord's method. The results can be seen in Table II and the accuracy of computations \( \varepsilon = 10^{-6} \). One iteration corresponds to the solution of one non-linear equation and one Cauchy's problems (for finding values of function) in the first and third cases, and to that of a non-linear equation and two Cauchy's problems (for finding values of function and their derivatives) in the second case, which all leads to additional difficulties.
### TABLE II  Different results corresponding to different methods by which system 5 is solved

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<th>n. of variant</th>
<th>( \sigma_0 )</th>
<th>( \lambda_1^{(0)} )</th>
<th>( \lambda_2^{(0)} )</th>
<th>number of iter.</th>
<th>( \lambda^{(0)} )</th>
<th>number of iter.</th>
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References