On the zeros of Bloch functions

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Abstract

A function $f$, analytic in the unit disc $\Delta$, is said to be a Bloch function if
$$\sup_{z \in \Delta} (1-|z|^2)|f'(z)| < \infty.$$ In this paper we study the zero sequences of non-trivial Bloch functions. Among other results we prove that if $f$ is a Bloch function with $f(0) \neq 0$ and $\{z_k\}$ is the sequence of ordered zeros of $f$, then
$$\prod_{k=1}^{N} \frac{1}{|z_k|} = O((\log N)^{1/2}), \quad \text{as } N \to \infty$$

and
$$\sum_{|z_k| > 1 - \frac{1}{e}} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-\alpha} < \infty, \quad \text{for all } \alpha > 1.$$

We will also prove that (ii) is best possible even for the little Bloch space $B_0$. To this end we construct a function $f \in B_0$ whose zero sequence $\{z_k\}$ satisfies
$$\sum_{|z_k| > 1 - \frac{1}{e}} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-1} = \infty.$$

We also consider analogous problems for some other related spaces of analytic functions.

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Let $\Delta$ denote the unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$. If $0 < r < 1$ and $g$ is a function which is analytic in $\Delta$, we set
\[
M_p(r,g) = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < p < \infty,
\]
\[
M_\infty(r,g) = \max_{|z|=r} |g(z)|.
\]
For $0 < p < \infty$, the Hardy space $H^p$ consists of those functions $f$ analytic in $\Delta$ for which
\[
\|f\|_{H^p} = \sup_{0 < r < 1} M_p(r,f) < \infty.
\]
A function $f$ analytic in $\Delta$ is said to be a Bloch function if
\[
\|f\|_B = |f(0)| + \sup_{z \in \Delta} (1 - |z|^2)|f'(z)| < \infty.
\]

The space of all Bloch functions will be denoted by $B$. Clearly, we have $H^\infty \subset B$.

We mention [1] as a general reference for the theory of Bloch functions.

In this paper we shall be mainly concerned with the zeros of Bloch functions. It is well known (see [3], chapter 2) that a sequence $\{ z_k \}$ is the zero set of an $H^p$-function, $0 < p \leq \infty$, if and only if it satisfies the Blaschke condition
\[
\sum_{k=1}^\infty (1 - |z_k|) < \infty.
\]

The Bergman space $A^p$, $0 < p < \infty$, is the class of functions $f$ analytic in $\Delta$ for which $|f(z)|$ belongs to $L^p(\Delta)$. Given $f \in A^p$, we define its $A^p$-norm as follows
\[
\|f\|_{A^p} = \left( \frac{1}{\pi} \int_\Delta |f(z)|^p \, dx \, dy \right)^{1/p}.
\]
It is clear that $H^p \subset A^p$ for all $p > 0$. The problem of describing the zero sets of $A^p$-functions has been extensively studied in [8] and [9].

If $f$ is an analytic function in $\Delta$, $f \neq 0$, and $\{ z_k \}_{k=1}^\infty$ is the sequence of its zeros, repeated according to multiplicity and ordered so that $|z_1| \leq |z_2| \leq |z_3|, \ldots$, then $\{ z_k \}$ is said to be the sequence of ordered zeros of $f$.

Among other results, let us recall that of Horowitz [8]. He proved that if $f$ is an $A^p$-function, $0 < p < \infty$, with $f(0) \neq 0$ and $\{ z_k \}$ is its sequence of ordered zeros then
\[
\prod_{k=1}^N \frac{1}{|z_k|} = O(N^{1/p}), \quad \text{as } N \to \infty. \tag{1.1}
\]

Horowitz also proved (see [8, corollary 4.11]), that the exponent $1/p$ cannot be decreased in (1.1). However, our first result shows that $O(N^{1/p})$ can be replaced by $o(N^{1/p})$.

**Theorem 1.** Let $0 < p < \infty$ and let $f \in A^p$ with $f(0) \neq 0$. Let $\{ z_k \}_{k=1}^\infty$ be the sequence of ordered zeros of $f$. Then
\[
\prod_{k=1}^N \frac{1}{|z_k|} = o(N^{1/p}), \quad \text{as } N \to \infty. \tag{1.2}
\]
Let us define $A^0$ as the space of all functions $f$ analytic in $\Delta$ and such that
\[ M_\infty(r, f) = O\left(\log \frac{1}{1 - r}\right), \quad \text{as } r \to 1. \]

Obviously
\[ \mathcal{B} \subset A^0 \subset \bigcap_{0 < p < \infty} A^p. \]

The function $f(z) = (\log(1/1 - z))^2$ belongs to $\bigcap_{0 < p < \infty} A^p$, but not to $A^0$. Hence, the inclusion $A^0 \subset \bigcap_{0 < p < \infty} A^p$ is strict. We shall see later that the inclusion $\mathcal{B} \subset A^0$ is also strict.

Our next result is an analogue of (1·1) for the spaces $\mathcal{B}$ and $A^0$.

**Theorem 2.** Let $f$ be an analytic function in $\Delta$ with $f(0) \neq 0$ and let $\{z_k\}_{k=1}^\infty$ be the sequence of ordered zeros of $f$.

(i) If $f \in A^0$, then
\[ \prod_{k=1}^N \frac{1}{|z_k|} = O(\log N), \quad \text{as } N \to \infty. \tag{1·3} \]

(ii) If $f \in \mathcal{B}$, then
\[ \prod_{k=1}^N \frac{1}{|z_k|} = O((\log N)^{\frac{1}{2}}), \quad \text{as } N \to \infty. \tag{1·4} \]

We do not know whether or not (1·4) is sharp for the space of Bloch functions $\mathcal{B}$. However (1·3) is sharp for the space $A^0$. Indeed, we can prove that $O(\log N)$ cannot be replaced by $o(\log N)$ in (1·3).

**Theorem 3.** There exists a function $f \in A^0$ with $f(0) \neq 0$ whose ordered zeros $\{z_k\}$ satisfy
\[ \prod_{k=1}^N \frac{1}{|z_k|} = o(\log N), \quad \text{as } N \to \infty. \tag{1·5} \]

We note that if $f$ is the function given in Theorem 3, then it does not satisfy (1·4) and hence, Theorem 2 implies $f \notin \mathcal{B}$. This shows that, as mentioned above, the inclusion $\mathcal{B} \subset A^0$ is strict.

Two important subspaces of the space of Bloch functions $\mathcal{B}$ are $\mathcal{B}_0$ and $\mathcal{B}_1$. The space $\mathcal{B}_0$ consists of those $f \in \mathcal{B}$ for which
\[ (1 - |z|)|f'(z)| \to 0, \quad \text{as } |z| \to 1. \]
Equivalently, $\mathcal{B}_0$ is the closure of polynomials in the Bloch norm (cf. [1, theorem 2·1]).

The space $\mathcal{B}_1$ consists of those $f \in \mathcal{B}$ for which the following condition is satisfied:

If $\{z_n\} \subset \Delta$ and $|f(z_n)| \to \infty$, then $(1 - |z_n|)|f'(z_n)| \to 0$.

Clearly,
\[ \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}. \tag{1·6} \]

If $f(z) = \sum_{n=0}^\infty a_n z^n \in \mathcal{B}_0$, then $a_n \to 0$. By contrast, Fernández [4, 5] constructed functions in $\mathcal{B}_1$ whose coefficients do not tend to zero. Important examples of Bloch
functions are given by power series with Hadamard gaps, that is, power series of the form
\[ f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \] analytic in \( \Delta \) with \( n_{k+1} \geq \lambda n_k \), for all \( k \) and some constant \( \lambda > 1 \).

For such an \( f \) we have [1, p. 19] \( f \in \mathcal{B} \) if and only if \( \sup |a_k| < \infty \) and then by [1, 13]
\[ f \in \mathcal{B}_0 \iff f \in \mathcal{B}_1 \iff a_k \to 0. \]

With these results in mind, it is clear that the inclusions in (1·6) are strict.

Our next result shows that (1·3) can be improved for \( \mathcal{B}_1 \)-functions.

**Theorem 4.** Let \( f \in \mathcal{B}_1 \) with \( f(0) \neq 0 \) and let \( \{z_k\}_{k=1}^{\infty} \) be the sequence of ordered zeros of \( f \). Then
\[ \prod_{k=1}^{N} \frac{1}{|z_k|} = o((\log N)^1), \quad \text{as } N \to \infty. \quad (1·7) \]

The zeros of a Bloch function need not satisfy the Blaschke condition. Anderson, Clunie and Pommerenke proved in [1] that if \( f \) is a Bloch function, \( D \) is a disc that touches \( \partial \Delta \) from inside and \( \{z_n\} \) is the sequence of zeros of \( f \) then
\[ \sum_{z_n \in D} (1 - |z_n|) < \infty. \]

Next we consider the question of finding a substitute of the Blaschke condition valid for the zero sequences of Bloch functions. We can prove the following results.

**Theorem 5.** Let \( f \in A^0, f \neq 0 \), and let \( \{z_k\}_{k=1}^{\infty} \) be the ordered sequence of zeros of \( f \). Then
\[ \sum_{|z_k| > 1 - \frac{1}{\alpha}} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-\alpha} < \infty, \quad (1·8) \]
for all \( \alpha > 1 \).

Since \( \mathcal{B} \subset A^0 \) (1·8) also holds if \( f \in \mathcal{B} \) and \( \alpha > 1 \). Next, we shall show that the conclusion of Theorem 5 is best possible even for the space \( \mathcal{B}_0 \).

**Theorem 6.** There exists a function \( f \in \mathcal{B}_0, f \neq 0 \), whose ordered zeros \( \{z_k\}_{k=1}^{\infty} \) satisfy
\[ \sum_{|z_k| > 1 - \frac{1}{\alpha}} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-1} = \infty. \quad (1·9) \]

The proofs of Theorems 1, 2, 3 and 4 will be presented in Section 2. The proofs of Theorems 5 and 6 will be included in Section 3. In what follows we assume that \( C \) denotes a positive constant (which may depend on \( p \) and \( f \) but not on \( r, N, n, k, \ldots \)) and which may be different at each occurrence.
Our proofs of Theorems 1, 2 and 4 will be based on the following result due to Horowitz.

**Lemma A** [8, p. 695]. Let \( f \) be an analytic function in \( \Delta \) with \( f(0) \neq 0 \) and let \( \{z_k\} \) be the sequence of its ordered zeros. Then for \( 0 < p < \infty \), for \( 0 \leq r < 1 \) and for all positive integers \( N \),

\[
|f(0)|^p \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \leq M_p(r, f)^p. \tag{2.1}
\]

It is worth pointing out that Lemma A is strongest when we choose \( N \) with \( |z_n| < r \). The result for larger values of \( N \) is a simple consequence of this case.

**Proof of Theorem 1.** Let \( 0 < p < \infty \) and let \( f \in A^p \) with \( f(0) \neq 0 \). Let \( \{z_k\}_{k=1}^{\infty} \) be its sequence of ordered zeros. If \( f \in A^0 \) then, clearly,

\[
M_1(r, f) = O\left(\log \frac{1}{1-r}\right), \quad \text{as } r \to 1.
\]

Using Lemma A with \( p = 1 \) and \( r = 1 - (1/N) \), we have

\[
|f(0)| \left(1 - \frac{1}{N}\right)^N \prod_{n=1}^{N} \frac{1}{|z_n|} \leq C (\log N).
\]

This gives (1.3).

If \( f \in \mathcal{H} \), then according to [2] and [10] (see also [13, p. 186]) we have for \( 0 < p < \infty \),

\[
M_p(r, f) = O\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{p}}\right), \quad \text{as } r \to 1.
\]
Using this with $p = 1$, and Lemma A with $p = 1$ and $r = 1 - (1/N)$, we obtain

$$|f(0)| \left(1 - \frac{1}{N}\right)^N \prod_{n=1}^{N} \frac{1}{|z_n|} \leq C (\log N)^{\frac{1}{2}}.$$ 

This gives (1·4) and completes the proof of Theorem 2.

Girela proved in [6, theorem 1] that, for $f \in B_1$ and $0 < p < \infty$,

$$M_p(r, f) = o\left(\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}\right), \quad \text{as } r \to 1.$$ 

Using this with $p = 1$, and Lemma A with $p = 1$ and $r = 1 - (1/N)$, we obtain

$$|f(0)| \left(1 - \frac{1}{N}\right)^N \prod_{n=1}^{N} \frac{1}{|z_n|} = o((\log N)^{\frac{1}{2}}),$$

which implies (1·7) and Theorem 4 is proved.

**Proof of Theorem 3.** The reasoning we are going to apply in our proof of Theorem 3 is related to that used by Horowitz in [9, p. 330].

Let

$$f(z) = \prod_{k=1}^{\infty} F_k(z), \quad z \in \Delta, \quad (2·4)$$

where,

$$F_k(z) = \frac{1 + e^{\frac{i}{k} z^{2k}}}{1 + e^{-\frac{i}{k} z^{2k}}}, \quad z \in \Delta, \quad k = 1, 2, \ldots. \quad (2·5)$$

It is clear that the product in (2·4) converges uniformly on every compact subset of $\Delta$ and so, it defines a function $f$ which is analytic in $\Delta$ and the set of its zeros is the union of the zero sets of the $F_k$s. Hence, for every $k = 1, 2, \ldots$, the function $f$ has exactly $2^k$ zeros on the circle $\{|z| = e^{-\frac{1}{k}}\}$. If $N_n = 2^{1} + 2^{2} + \cdots + 2^{n}$, then

$$\prod_{k=1}^{N_n} \frac{1}{|z_k|} = \prod_{k=1}^{n} e^{1/k} = e^{1 + \frac{1}{2} + \cdots + \frac{1}{n}} = e^{\log n + O[1]} \geq Cn. \quad (2·6)$$

Since $2^n \leq N_n \leq 2^{n+1}$, we have $\log N_n \sim n$. Therefore (2·6) implies

$$\prod_{n=1}^{N} \frac{1}{|z_n|} = o(\log N).$$

Hence, $f$ satisfies (1·5).

Now we turn to prove that $f \in A^0$. Set

$$r_n = e^{-\frac{1}{n}}, \quad n = 1, 2, \ldots. \quad (2·7)$$

We have

$$|f(z)| = \prod_{k=1}^{n} \frac{e^{\frac{i}{k} z^{2k}} + z^{2k}}{1 + e^{-\frac{i}{k} z^{2k}}} \prod_{j=1}^{\infty} \frac{1 + e^{\frac{i}{j} z^{2j}}}{1 + e^{-\frac{i}{j} z^{2j}}}. \quad (2·8)$$
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Using the inequality \(|(a + b)/(1 + ab)| \leq (|a| + |b|)/(1 + |ab|) < 1, \ (|a|, |b| < 1)|, we get

\[
\left| \frac{e^{-\frac{1}{2}z^2} + z^2}{1 + e^{-\frac{1}{2}z^2}} \right| < 1, \quad z \in \Delta, \quad k = 1, 2, \ldots
\]  

and

\[
\left| \frac{1 + e^{\frac{1}{2n}z^{2^{2n-j}}}}{1 + e^{-\frac{1}{2n}z^{2^{2n-j}}}} \right| = e^{\frac{1}{2n}} \left| \frac{e^{-\frac{1}{2n}z^{2^{2n-j}}} + z^{2^{2n-j}}}{1 + e^{-\frac{1}{2n}z^{2^{2n-j}}}} \right| \leq e^{\frac{1}{2n}} \left| \frac{e^{-\frac{1}{2n}z^{2^{2n-j}}} + (1/e)^{2^j}}{1 + e^{\frac{1}{2n}(1/e)^{2^j}}} \right| \leq \frac{1 + e^{\frac{1}{2n}(1/e)^{2^j}}}{1 + e^{-\frac{1}{2n}(1/e)^{2^j}}}, \quad |z| = r_n, \quad j \geq 1, \quad n \geq 1. \tag{2.10}
\]

Using (2.10) and the fact that \(x \mapsto (1 + x\alpha)/(1 + x^{-1}\alpha)\) is increasing in \((0, 1]\), for a constant \(\alpha > 0\), we deduce that

\[
\prod_{j=1}^{\infty} \left| \frac{1 + e^{\frac{1}{2n}z^{2^{2n-j}}}}{1 + e^{-\frac{1}{2n}z^{2^{2n-j}}}} \right| \leq \prod_{j=1}^{\infty} \left| \frac{1 + e^{\frac{1}{2n}(1/e)^{2^j}}}{1 + e^{-\frac{1}{2n}(1/e)^{2^j}}} \right| \leq \prod_{j=1}^{\infty} \frac{1 + e^{(1/e)^{2^j}}}{1 + e^{-1(1/e)^{2^j}}} = A, \quad |z| = r_n, \quad n \geq 1, \tag{2.11}
\]

where \(A\) is independent of \(n\).

Using (2.8), (2.9) and (2.11), we see that

\[
|f(z)| \leq A \prod_{k=1}^{n} e^{1/k} = Ae^{1/2 + \cdots + 1/k} = Ae^{\log n + O(1)} \leq Cn, \quad |z| = r_n, \quad n \geq 1. \tag{2.12}
\]

Notice that for \(|z| = r_n\)

\[
2^{2n} = \frac{1}{\log (1/|z|)} \leq \frac{1}{1 - |z|}
\]

and then (2.12) gives

\[
|f(z)| \leq C \log \frac{1}{\log (1/|z|)} \leq C \log \frac{1}{1 - |z|}, \quad |z| = r_n, \quad n \geq 1. \tag{2.13}
\]

Now, if \(r_n \leq |z| \leq r_{n+1}\) and \(n\) is sufficiently large we deduce, using (2.13), that

\[
|f(z)| \leq M_\infty(r_{n+1}, f) \leq C \log \frac{1}{\log (1/r_{n+1})} = C \log (2^{n+1}) = Cn \leq C \log \frac{1}{1 - r_n} \leq C \log \frac{1}{1 - |z|}.
\]

Hence, \(f \in A^0\). This finishes the proof.
Before embarking into the proof of Theorem 5, we recall some notation and facts from Nevanlinna theory (see [7, 11 or 14]). For simplicity, we shall restrict ourselves to analytic functions in $\Delta$.

Let $f$ be a function analytic and not constant in $\Delta$. For any $a \in \mathbb{C}$ and $0 \leq r < 1$, we denote by $n(r,a,f)$ the number of zeros of $f - a$ in the disc $\{|z| \leq r\}$, where each zero is counted according to its multiplicity. We define also

$$N(r,a,f) = \int_0^r \frac{n(t,a,f) - n(0,a,f)}{t} \, dt + n(0,a,f) \log r, \quad 0 < r < 1. \quad (3\cdot1)$$

For simplicity, we shall write

$$n(r,f) = n(r,0,f), \quad N(r,f) = N(r,0,f).$$

The Nevanlinna characteristic function $T(r,f)$ is defined by

$$T(r,f) = \frac{1}{2\pi} \int_{-\pi}^\pi \log^+ |f(re^{i\theta})| \, d\theta, \quad 0 < r < 1. \quad (3\cdot2)$$

The proximity function $m(r,a,f)$ is given by

$$m(r,a,f) = \frac{1}{2\pi} \int_{-\pi}^\pi \log^+ \frac{1}{|f(re^{i\theta}) - a|} \, d\theta, \quad 0 < r < 1. \quad (3\cdot3)$$

In this setting the first fundamental theorem can be stated as follows:

**First Fundamental Theorem of Nevanlinna.** Let $f$ be an analytic and not constant function in the unit disc $\Delta$. Then

$$m(r,a,f) + N(r,a,f) = T(r,f) + O(1), \quad \text{as} \quad r \to 1, \quad (3\cdot4)$$

for every $a \in \mathbb{C}$.

If $f \in \mathcal{B}$ (actually, even if simply $f \in A^0$), then clearly

$$T(r,f) = O\left(\log \log \frac{1}{1-r}\right), \quad \text{as} \quad r \to 1,$$

and using (3-4), we deduce that

$$N(r,a,f) = O\left(\log \log \frac{1}{1-r}\right), \quad \text{as} \quad r \to 1, \quad \text{for all} \quad a \in \mathbb{C}, \quad (3\cdot5)$$

which implies (see [1, p. 22])

$$n(r,a,f) = O\left(\frac{1}{1-r} \log \log \frac{1}{1-r}\right), \quad \text{as} \quad r \to 1, \quad \text{for all} \quad a \in \mathbb{C}. \quad (3\cdot6)$$

Now, we can proceed to prove our results.

**Proof of Theorem 5.** Let $f \in A^0$ and let $\{z_k\}_{k=1}^\infty$ be the sequence of ordered zeros of $f$. Let us assume, without loss of generality, that

$$1 - \frac{1}{e} < |z_k|, \quad \text{for all} \quad k.$$
Using Theorem 2, we deduce that

\[
\sum_{k=1}^{N} \log \frac{1}{|z_k|} \leq \log \log N + O(1), \tag{3-7}
\]

Now \((1 - |z_k|) \leq \log(1/|z_k|)\) and then (3-7) implies

\[
\sum_{k=1}^{N} (1 - |z_k|) \leq \log \log N + O(1). \tag{3-8}
\]

Since \(|z_k|\) is increasing, (3-8) implies

\[N(1 - |z_N|) \leq \log \log N + O(1)\]

and hence

\[
\frac{N}{\log \log N} \leq \frac{2}{(1 - |z_N|)}, \quad \text{if } N \geq N_0, \tag{3-9}
\]

for a certain \(N_0 > 1\). Using (3-9) and adding by parts, we obtain

\[
\sum_{k=N_0}^{N} (1 - |z_k|) \left( \log \log \frac{1}{1 - |z_k|} \right)^{-\alpha} \leq C \sum_{k=N_0}^{N} (1 - |z_k|) (\log \log k)^{-\alpha}
\]

\[
= \sum_{k=N_0}^{N-1} \left( \sum_{j=N_0}^{k} (1 - |z_j|) \right) [ (\log \log k)^{-\alpha} - (\log \log (k+1))^{-\alpha} ]
\]

\[
+ \sum_{j=N_0}^{N} (1 - |z_j|) (\log \log N)^{-\alpha}
\]

\[
= I + II.
\]

Now, using (3-8) and having in mind that \(\alpha > 1\), we obtain

\[I \leq \sum_{k=N_0}^{N-1} \frac{\log \log k}{k \log k (\log \log k)^{-\alpha}} = O(1)\]

and

\[II = O((\log \log N)^{1-\alpha}) = O(1)\]

Hence, we obtain (1-8). This finishes the proof.

**Proof of Theorem 6.** Set

\[
g(z) = \sum_{n=1}^{\infty} \frac{1}{1 + \log n} z^n, \quad z \in \Delta. \tag{3-10}
\]

Since \(g\) is given by a power series with Hadamard gaps whose coefficients tend to zero, it follows that \(g \in \B_0\). Set

\[r_n = 1 - 2^{-n}, \quad n = 1, 2, 3, \ldots \tag{3-11}\]
We have, for all sufficiently large $n$,

$$M_2(r_n, g)^2 = \sum_{k=1}^{\infty} \frac{1}{(1 + \log k)^2 \frac{r_n}{n}} \left(1 + \log \left(1 + \log k\right) + \frac{1}{2^n}\right)^{2n+1} \geq n \sum_{k=1}^{n} \frac{1}{(1 + \log k)^2 \frac{r_n}{n}} \left(1 + \frac{1}{2^n}\right)^{2n+1} \geq C \frac{n}{(\log n)^2}. \quad (3.12)$$

Now, since $\log \left(\frac{1}{1 - r_n}\right) = n \log 2$, (3.12) shows that

$$M_2(r_n, g)^2 \geq C \log \left(\frac{1}{1 - r_n}\right)^{-2} \left(\log \log \left(\frac{1}{1 - r_n}\right)\right)^{-2}, \quad \text{if } n \text{ is sufficiently large.} \quad (3.13)$$

From this we can deduce easily that

$$M_2(r, g)^2 \geq C \log \left(\frac{1}{1 - r}\right)^{-2} \left(\log \log \left(\frac{1}{1 - r}\right)\right)^{-2}, \quad \text{if } r \text{ is sufficiently close to 1.} \quad (3.14)$$

Since $g$ is given by a power series with Hadamard gaps, using theorem 8.25 in chapter V of [16, vol. I], we see that there exist two absolute constants $\lambda > 0$ and $\mu > 0$ such that for every $r \in (0, 1)$ the set

$$E_r = \{\theta \in [0, 2\pi]: |g(re^{i\theta})| > \lambda M_2(r, g)\}$$

has measure not less than $\mu$. This and (3.14) imply

$$T(r, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |g(re^{i\theta})| \, d\theta \geq \frac{1}{2\pi} \int_{E_r} \log^+ \left[C \left(\log \left(\frac{1}{1 - r}\right)\right)^{\frac{1}{2}} \left(\log \log \left(\frac{1}{1 - r}\right)\right)^{-1}\right] \, d\theta \geq \frac{\mu}{2\pi} \left[C + \frac{1}{2} \log \log \left(\frac{1}{1 - r}\right) - \log \log \left(\frac{1}{1 - r}\right)\right],$$

for all $r$ sufficiently close to 1. Hence if we take $\alpha, 0 < \alpha < \mu/4\pi$, we have

$$T(r, g) \geq \alpha \log \log \left(\frac{1}{1 - r}\right), \quad r_0 < r < 1, \quad (3.15)$$

for a certain $r_0 \in (0, 1)$.

Using (3.4), we see that for every $a \in \mathbb{C}$, (3.15) remains true with $T(r, f)$ replaced either by $m(r, a, f)$ or $N(r, a, f)$. This is not enough for our purposes. Applying some more refined results of Nevanlinna Theory (cf. the theorem on p. 276 of [11]) we see that (3.15) implies the existence of complex numbers $a$ such that $g(0) \neq a$ and

$$N(r, a, g) \geq \beta \log \log \left(\frac{1}{1 - r}\right), \quad r_0 < r < 1, \quad (3.16)$$

for some $\beta > 0$ and some $r_0 \in (0, 1)$. Take such an $a$ and set

$$f(z) = g(z) - a, \quad z \in \Delta. \quad (3.17)$$
On the zeros of Bloch functions

We have \( f \in \mathcal{B}_0 \), \( f(0) \neq 0 \), and (3.16) can be written as

\[
N(r, f) \geq \beta \log \log \frac{1}{1 - r}, \quad r_0 < r < 1.
\]  \tag{3.18}

Let \( \{z_n\} \) be the sequence of zeros of \( f \). We shall prove that the conclusion of Theorem 6 holds for this function \( f \). For simplicity, set \( n(r, f) = n(r) \) and \( N(r, f) = N(r) \). We have, integrating by parts and using (3-6),

\[
\sum_{|z_n| > 1 - \frac{1}{4}} (1 - |z_n|) \left( \log \log \frac{1}{1 - |z_n|} \right)^{-1} \geq \int_{r_0}^1 \frac{1 - r}{(\log \log (1/1 - r)) \log (1/1 - r)} dn(r) + O(1)
\]

\[
= \int_{r_0}^1 r \left( \log \log \frac{1}{1 - r} \right)^{-1} \left( 1 + \left( \log \log \frac{1}{1 - r} \right)^{-1} \left( \log \frac{1}{1 - r} \right)^{-1} \right) n(r) \frac{dr}{r} + O(1)
\]

\[
\geq \int_{r_0}^1 r \left( \log \log \frac{1}{1 - r} \right)^{-1} n(r) \frac{dr}{r} + O(1).
\]  \tag{3.19}

Having in mind (3.5), another integration by parts gives

\[
\int_{r_0}^1 r \left( \log \log \frac{1}{1 - r} \right)^{-1} \frac{n(r)}{r} dr = \int_{r_0}^1 \left( - \left( \log \log \frac{1}{1 - r} \right)^{-1} \right.

\[
+ \frac{r}{1 - r} \left( \log \log \frac{1}{1 - r} \right)^{-2} \left( \log \frac{1}{1 - r} \right)^{-1} \big) N(r) dr + O(1). \]  \tag{3.20}

Using (3.16), (3.19) and (3.20) we obtain

\[
\sum_{|z_n| > 1 - \frac{1}{4}} (1 - |z_n|) \left( \log \log \frac{1}{1 - |z_n|} \right)^{-1}
\]

\[
\geq C \int_{r_0}^1 \frac{r}{1 - r} \left( \left( \log \log \frac{1}{1 - r} \right)^{-1} \left( \log \frac{1}{1 - r} \right)^{-1} \big) dr + O(1)
\]

\[
= \infty.
\]

This finishes the proof of Theorem 6.

Remark. The existence of a function \( f \in \mathcal{B} \) satisfying (3.18) follows from the work of Offord \cite{12} in which probabilistic arguments were used (see also \cite{1}, p. 20). Our function belongs to \( \mathcal{B}_0 \) and, furthermore, its construction is much simpler.

REFERENCES


