A link between the bounds on relativistic velocities and areas of hyperbolic triangles

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In this paper we show the mathematical equivalence between two well-known facts: the existence of an upper bound for the area in Lobachevskian geometry and the existence of a limit for relativistic velocities. The key point is that the space of relativistic velocities can be interpreted as a Lobachevskian space.
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I. INTRODUCTION

The origin of this paper was a talk between the authors during a break between classes. One of the authors (N.A.) had just been teaching the students in her geometry course the remarkable fact that in Lobachevskian geometry the area of a triangle cannot be as large as one could wish, but is bounded by $-\pi/K$, where K is the negative constant curvature of the considered Lobachevskian space. The other author (C.C.) had been showing the students in his physics course the no less important fact that the relativistic velocity between two inertial observers cannot be indefinitely large, but is limited by c, the velocity of light in vacuum. We asked ourselves if there was any relation between these two facts. The surprising answer we have found is that they are equivalent for a suitable interpretation of the Lobachevskian space as the space of the relativistic velocities. Once again, the ''unreasonable effectiveness'' of mathematics appears that so impressed Eugene Wigner.[1]

II. POINCARÉ DISK AND RELATIVISTIC VELOCITIES

Let us consider three rocketships $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ that start moving away from a rocketship base $\mathcal{B}$. These rocketships are assumed to move radially in a plane, henceforth *OXY*, with uniform velocities v_1, v_2, and v_3, respectively, and forming arbitrary angles between them. Their world lines are semirays OP_1, OP_2, and OP_3 with slopes $1/v_1$, $1/v_2$, and $1/v_3$, respectively, with respect to the plane *OXY* in the Minkowski diagram of the inertial frame $K_{\mathcal{B}}$ associated with $\mathcal{B}$ (Fig. 1).

The position of these rocketships with respect to $K_{\mathcal{B}}$, when each one of them has spent a proper time of $\tau=1$ s, corresponds to the intersection of their world lines with the upper sheet H^+ of the hyperboloid of the equation $t^2-x^2-y^2=1$, where we have considered the light velocity c to be equal to 1.

In the simultaneity space for $t=1$ of the observer $K_{\mathcal{B}}$, the rocketships are at points Q_1, Q_2, and Q_3, which are the vertices of a triangle (Fig. 1). If O' is the position of the rocketship base in this diagram, we have

$$|O'Q_i|=\tanh\chi_i=v_i, \tag{1}$$

where χ_i is the distance on the hyperboloid between O' and P_i for $i=1,2,3$, see Fig. 2. This distance on the hyperboloid is that induced by the Lorentz metric $ds^2=dt^2-dx^2-dy^2$ on the Minkowski space. In the pseudospherical coordinates of the hyperboloid (χ,φ), the induced metric takes the form[2]

$$dl^2=d\chi^2+\sinh^2\chi\, d\varphi^2.$$

The intrinsic curvature of the hyperboloid provided with this metric is a constant equal to -1, and therefore the hyperboloid is a model of a bidimensional Lobachevskian space (or hyperbolic space) which is sometimes called the pseudosphere. Taking the previous considerations into account we shall see at the end of this section that the upper sheet H^+ of the hyperboloid represents the space of relativistic velocities.[3]

Another representation of the relativistic velocities space is the disk K determined by the intersection of the light cone and the simultaneity plane $t=1$ (Fig. 1). This disk, with the metric induced on it by the stereographic projection from O of H^+, constitutes another model of a bidimensional Lobachevskian space known as the Klein model. However, this model is not conformal,[4] that is, the angles do not have the same meaning as in the Euclidean case. We can obtain a conformal model on the disk $\mathcal{P}$ of the plane *OXY* centered in O and of radius 1. For this, let us consider the stereographic projection of H^+,f, from the point S of coordinates (0, 0, -1) (the ''south pole'' of the pseudosphere), onto $\mathcal{P}$ (Fig. 3). The disk $\mathcal{P}$ with the metric induced through the map f by the metric of the hyperboloid is a conformal bidimensional model of a Lobachevskian space which is known as the Poincaré disk.[4–6]

Therefore the Lobachevskian distance $|OR|_L$ from O to $f(P)=R$, in the disk $\mathcal{P}$, is equal to the distance χ in the hyperboloid from O' to P. From Eq. (1) we find that the velocity v and the Lobachevskian distance on the disk $\mathcal{P}$ are related by

$$v=\tanh|OR|_L. \tag{2}$$

Note that for the Euclidean distance on the disk $\mathcal{P}$ we have

$$|OR|_E=\tanh\frac{|OR|_L}{2}.$$

So, we have the following relation between these two distances:

$$|OR|_L=\ln\frac{1+|OR|_E}{1-|OR|_E}.$$

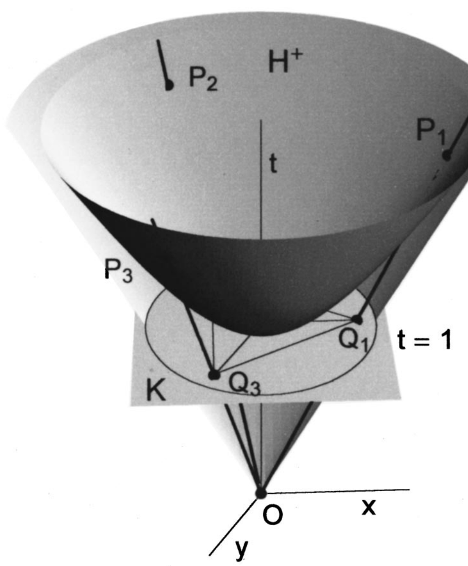

Fig. 1. Word lines of three rocketships moving away radially from a rocketship base, with uniform rectilinear velocities. P_1, P_2, and P_3 represent the position of the rocketships on the hyperboloid corresponding to proper time $\tau=1$. Q_1, Q_2, and Q_3 represent the position of the rocketships on the simultaneity space for $t=1$ of the rocketships' base. The hyperboloid H^+ and the disk K are models of the Lobachevskian geometry, and they can be interpreted as spaces of relativistic velocities.

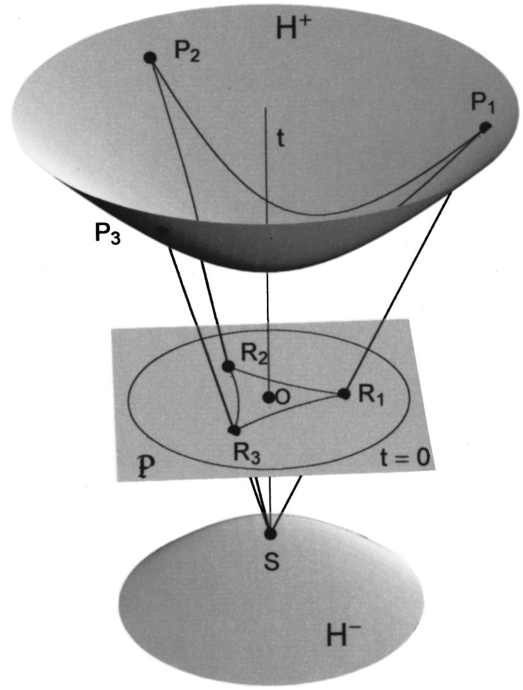

Fig. 3. Stereographic projection from the south pole S of the hyperboloid H^+ onto the Poincaré disk in the plane $t=0$. This disk is a conformal model of the space of the relativistic velocities. The hyperbolic triangle $P_1P_2P_3$ projects into the hyperbolic triangle $R_1R_2R_3$. The relative velocity between two rocketships is the tanh of the connecting side's length.

From this formula, we deduce that when R approaches the border of the disk, i.e., $|OR|_E \to 1$, the corresponding Lobachevskian distance tends to infinity, i.e., $|OR|_L \to \infty$, and the velocity approaches the velocity of light, i.e., $v = \tanh|OR|_L \to 1$.

The above distance on the disk $\mathcal{P}$ can be infinitesimally expressed by the metric tensor[2]

$$ds^2 = 4\,\frac{dr^2 + r^2\,d\varphi^2}{(1-r^2)^2}$$

in terms of the polar coordinates (r, φ).

In accordance with the above construction, the Poincaré disk $\mathcal{P}$ becomes the relativistic velocities space for the case of a bidimensional space. More precisely, it is a representation of the relativistic velocities space associated with the observer $K_{\mathcal{B}}$. Any other inertial observer $K_{\mathcal{R}}$ is represented by one point R of the disk, in such a way that $\tanh|OR|_L$ corresponds to the module of the velocity of $K_{\mathcal{R}}$ with respect to $K_{\mathcal{B}}$, and the direction of OR to the direction of the velocity of $K_{\mathcal{R}}$ with respect to $K_{\mathcal{B}}$. Observe that the whole class of inertial observers which are at rest with respect to $K_{\mathcal{R}}$ corresponds to the point R in the disk.

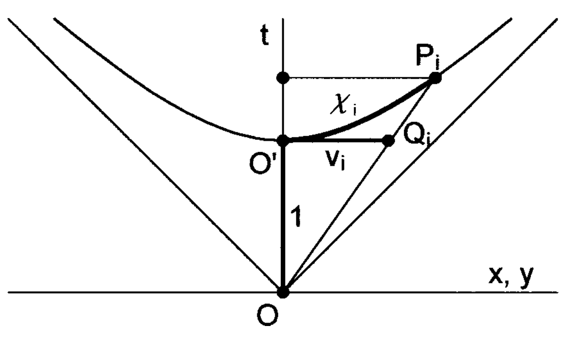

Fig. 2. Transverse section of Fig. 1 showing the Lobachevskian distance χ_i from O' to $P_i=(\sinh\chi_i, \cosh\chi_i)$ and the velocity $v_i=|O'Q_i|$ that is equal to $\tanh\chi_i$ from triangle proportionality.

For any inertial observer we can obtain a representation in the Poincaré disk of the relativistic velocities space in which that observer is represented by the center. To pass from one representation to another we only have to use the appropriate Lorentz transformation. For example, the Lorentz transformation that takes $K_{\mathcal{R}}$ into $K_{\mathcal{B}}$ is an isometry in the hyperboloid that takes the point $P=f^{-1}(R)$ into $O'=f^{-1}(O)$, and this transformation induces an isometry on the disk $\mathcal{P}$ that takes R into O.[7,8]

We can deduce from Eq. (2) the formula for the relative velocity $v_{\mathcal{S}|\mathcal{R}}$ of two inertial observers $K_{\mathcal{R}}$ and $K_{\mathcal{S}}$ in terms of the distance between the corresponding points R, S in $\mathcal{P}$:

$$v_{\mathcal{S}|\mathcal{R}} = \tanh|RS|_L. \tag{3}$$

In fact, for this it suffices to consider an isometry of the disk $\mathcal{P}$ that takes R into O, induced by a Lorentz transformation that takes $K_{\mathcal{R}}$ into $K_{\mathcal{B}}$.

To compose relativistic velocities of three inertial observers $K_{\mathcal{R}}$, $K_{\mathcal{S}}$, and $K_{\mathcal{T}}$ we proceed as follows. Since $v_{\mathcal{S}|\mathcal{R}} = \tanh|RS|_L$ and $v_{\mathcal{T}|\mathcal{S}} = \tanh|ST|_L$, then we can obtain $v_{\mathcal{T}|\mathcal{R}}$ by calculating in the hyperbolic triangle RST the length of the side RT from the other sides' lengths $|RS|_L = \tanh^{-1} v_{\mathcal{S}|\mathcal{R}}$, $|ST|_L = \tanh^{-1} v_{\mathcal{T}|\mathcal{S}}$, and the angle between the velocities $\mathbf{v}_{\mathcal{S}|\mathcal{R}}$ and $\mathbf{v}_{\mathcal{T}|\mathcal{S}}$. For this calculation we use the hyperbolic law of cosines.[9]

In particular, if $\mathbf{v}_{\mathcal{S}|\mathcal{R}}$ and $\mathbf{v}_{\mathcal{T}|\mathcal{S}}$ are collinear (i.e., $|RT|_L = |RS|_L + |ST|_L$), the formula for the tanh of the sum of the distances $|RS|_L$ and $|ST|_L$:

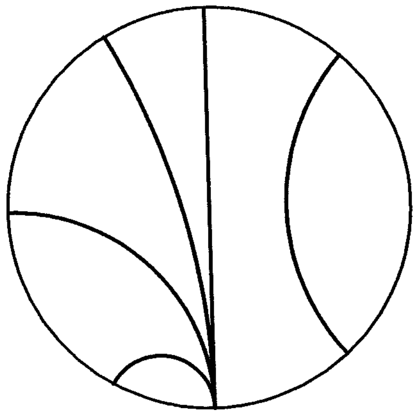

Fig. 4. Geodesics in the Poincaré disk.

$$\tanh|RT|_L=\frac{\tanh|RS|_L+\tanh|ST|_L}{1+\tanh|RS|_L\tanh|ST|_L},$$

leads to the usual formula:

$$v_{T|R}=\frac{v_{S|R}+v_{T|S}}{1+v_{S|R}v_{T|S}}.$$

This shows that the Poincaré disk (and therefore the hyperboloid H^+, the Klein disk, or any other isometric space) is a representation of the relativistic velocities space.[10]

III. THE AREA OF A HYPERBOLIC TRIANGLE AND RELATIVISTIC VELOCITIES

The geodesics in the hyperboloid H^+ are obtained[11] as intersections with the hyperboloid of planes passing through the origin O. Therefore, the geodesics in the Poincaré disk are obtained from these by the stereographic projection described above (see Fig. 3). These correspond to disk diameters and to circle arcs which orthogonally cut the border of the disk,[11] see Fig. 4.

In the hypothesis at the beginning of Sec. II, let us consider the stereographic projection of the hyperbolic triangle with vertices P_1, P_2, and P_3 in the hyperboloid H^+, which will correspond to a hyperbolic triangle in the Poincaré disk with vertices R_1, R_2, and R_3, see Figs. 3 and 5.

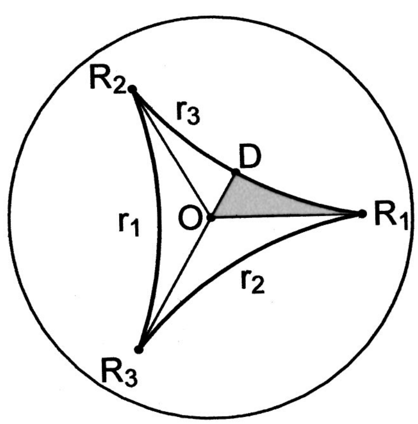

Fig. 5. The hyperbolic triangle $R_1R_2R_3$ corresponding to the rocketships' frames $K_{\mathcal{R}_1}$, $K_{\mathcal{R}_2}$, and $K_{\mathcal{R}_3}$ in the Poincaré disk. The angle $\alpha=\angle R_1=\angle R_2=\angle R_3$ can be given in terms of the velocity $v=\tanh|OR_i|_L$ by solving the right-angle triangle ODR_1.

If we denote by r_1, r_2, and r_3 the lengths of the triangle sides opposite to the vertices R_1, R_2, and R_3, we obtain, according to Eq. (3),

$$v_{\mathcal{R}_2|\mathcal{R}_1}=\tanh r_3,\quad v_{\mathcal{R}_3|\mathcal{R}_2}=\tanh r_1,\quad v_{\mathcal{R}_1|\mathcal{R}_3}=\tanh r_2,$$

and

$$v_{R_i|O}=v_i=\tanh|OR_i|_L,\quad i=1,2,3.$$

It is well known[12] that in a Lobachevski space with curvature $K=-1$ the area A_T of a triangle of vertices R_1, R_2, and R_3 coincides with the angular defect:

$$A_T=\pi-(\angle R_1+\angle R_2+\angle R_3). \tag{4}$$

In particular, this expression shows that the triangle area A_T is bounded by π.

We are now going to show the link between the bounds on relativistic velocities and areas of hyperbolic triangles. In order to simplify the proof, we choose the hyperbolic triangle with vertices R_1, R_2, and R_3 to be equilateral, which corresponds to a symmetrical situation of the rocketships with respect to the observer $K_{\mathcal{B}}$, that is:

$$v_1=v_2=v_3=v,\quad \angle R_1=\angle R_2=\angle R_3=\alpha.$$

In this case Eq. (4) yields

$$A_T=\pi-3\alpha. \tag{5}$$

Now, by using formulas of hyperbolic trigonometry, it is easy to solve the above triangle.

In fact, consider the right-angle triangle ODR_1, where D corresponds to the base of the perpendicular straight line from O to R_1R_2, see Fig. 5. Then we have

$$\beta=\angle O=\frac{\pi}{3},\quad \angle D=\frac{\pi}{2},\quad |OR_1|_L=d=\tanh^{-1}v.$$

So, if we use the formula[13]

$$\cot\beta\cot\frac{\alpha}{2}=\cosh d,$$

together with the fact that

$$\cosh d=\frac{1}{\sqrt{1-\tanh^2 d}}=\frac{1}{\sqrt{1-v^2}},$$

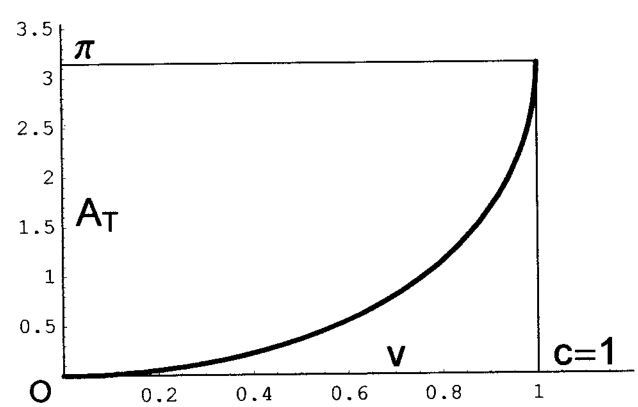

Fig. 6. The triangle area A_T as function of the velocity v.

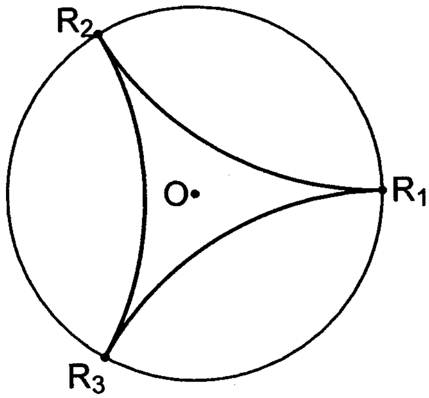

Fig. 7. In the limit triangle $R_1R_2R_3$ as v tends to infinity, the angle α is 0 and A_T is π.

we obtain

$$\cot\frac{\alpha}{2}=\frac{\sqrt{3}}{\sqrt{1-v^2}}. \tag{6}$$

By substituting α in Eq. (5), we obtain the relation between A_T and v:

$$A_T=\pi-6\arctan\frac{\sqrt{1-v^2}}{\sqrt{3}}. \tag{7}$$

Figure 6 gives a graphic representation of this function.

From Eq. (7) it follows that $A_T\to\pi$ whenever $v\to 1$, and vice versa.

Notice that in the Poincaré disk we approach the limit situation whenever the vertices R_1, R_2, and R_3 tend to the boundary, so that the angle α tends to zero and, according to Eq. (5), A_T tends to π (see Fig. 7).

This shows that the existence of an upper bound for the area of a triangle in Lobachevskian geometry is equivalent to the existence of a limit for the relative velocity between inertial frames.

Finally, observe that Eq. (6) can also be obtained from the relativistic velocity addition formula. In fact, let us consider first the case of the inertial frame $K_{\mathcal{R}_1}$ in which $\mathcal{R}_1$ is at rest and axis Y is parallel to the straight line joining $\mathcal{R}_2$ and $\mathcal{R}_3$. In this inertial frame, the angle α between the velocities $\mathbf{v}_{R_2|R_1}$ and $\mathbf{v}_{R_3|R_1}$ is given by

$$\tan\frac{\alpha}{2}=\frac{u_Y}{-u_X},$$

where (u_X,u_Y) are the components of the velocity $\mathbf{u}=\mathbf{v}_{R_2|R_1}$ of $\mathcal{R}_2$ with respect to $\mathcal{R}_1$.

On the other hand, in the inertial frame $K_{\mathcal{B}}$ in which $\mathcal{B}$ is at rest and the axes are parallel to the above, the velocity of $\mathcal{R}_2$ with respect to $\mathcal{B}$ is given by

$$\mathbf{u}'=(u'_X,u'_Y)=\left(-\frac{1}{2}v,\frac{\sqrt{3}}{2}v\right).$$

Now, by using the relativistic velocity addition formula[14] we obtain

$$\mathbf{u}=\frac{(u'_X-v,u'_Y\sqrt{1-v^2})}{1-u'_Xv}=\frac{(-3v,\sqrt{3}v\sqrt{1-v^2})}{2+v^2},$$

and then

$$\tan\frac{\alpha}{2}=\frac{u_Y}{-u_X}=\frac{\sqrt{1-v^2}}{\sqrt{3}},$$

which is, as expected, the same formula [see Eq. (6)] that we had obtained by mean of hyperbolic trigonometry.

IV. FINAL REMARKS

The previous results suggest to us an experiment analogous to that made by Gauss in order to determine if spherical geometry was preferable to Euclidean geometry for describing physical space.

Gauss[15] measured the angle sum of a triangle formed by the top of three hills, to verify if that sum was bigger than π.

For the problem we are concerned with, we can consider the triangle determined by three rocketships $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ moving as described after Eq. (4) in Sec. III. These rocketships play the role of the three hills in Gauss's experiment. From each one of them we would measure the angle formed by the light signals coming from the other two, which is equal to the angle formed by the velocities with respect to each rocketships of the other two.[16]

Therefore, we would verify that the sum of these angles is less than π and tends to zero as the velocities of the rocketships (with respect to the starting frame) tend to that of light.

However, we find a similar difficulty for this experiment as Gauss found for his. It consists in that the angular defect depends [see Eq. (5)] on the area of the triangle in such a way that for small triangles corresponding to small velocities, the area is small and the angle sum is approximately 180°, so that the geometry is indistinguishable from the Euclidean. For example, an angular defect of 1″ requires a velocity v of 579 km/s, as we deduce from Eq. (7).

Another question that we would like to raise is the following. It is well known that there exist only three kinds of geometries with constant curvature: the hyperbolic, with negative curvature; the Euclidean, with null curvature; and the spherical, with positive curvature. So far, the hyperbolic and the Euclidean geometries have physical models, namely, the space of relativistic velocities associated with the Lorentz transformation group, and the space of Newtonian velocities associated with the Galileo transformation group, respectively. So, the question is whether there exists a kinematics associated with the spherical geometry. This kinematics would not be incompatible with the principle of relativity. In fact, accepting this principle together with Euclidicity and isotropy, it is possible to prove[17] that there can exist only three possibilities, corresponding to the three geometries of constant curvature. The problem remains of whether there is a physical interpretation of the corresponding transformation group.

Finally, let us observe that in order to simplify the presentation we have restricted ourselves to considering only two spatial dimensions, but the same results hold for three spatial dimensions by interpreting the space of relativistic velocities as the three-dimensional Lobachevskian space.

ACKNOWLEDGMENTS

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[1]Eugene P. Wigner, ''The unreasonable effectiveness of mathematics in natural sciences,'' Commun. Pure Appl. Math. **13**, 1–14 (1960). A reprint of this famous essay can be found, for example, in *The Word Treasury of Physics, Astronomy, and Mathematics*, edited by Timothy Ferris (Little, Brown, Boston, 1991), pp. 526–540.

[2]B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry-Methods and Applications, Part I* (Springer-Verlag, New York, 1984), pp. 90–93.

[3]The German physicist Arnold Sommerfeld (1868–1951) was the person who first connected Lobachevskian geometry with special relativity. In his paper ''On the composition of velocities in relativity theory'' [Arnold Sommerfeld, ''Über die Zusammensetzung der Geschwindigkeiten in der Relativitätstheorie,'' Phys. Z. **10**, (22)826–829 (1909)] he established the relation between the formula for the addition of velocities in the theory of relativity and the trigonometric formulas for hyperbolic geometry. But it was the Yugoslav geometer Vladimir Varichak in the paper ''On the non-Euclidean interpretation of the theory of relativity'' [Vladimir Varichak, ''Über die nichteuklidische Interpretation der Relativitätstheorie,'' Jahrb. Deut. Math. Verein **21**, 103–122 (1912)], who pointed out that those formulas were formulas of Lobachevskian geometry. These and other historical notes can be found in B. A. Rosenfeld, *A History of Non-Euclidean Geometry* (Springer-Verlag, New York, 1988), pp. 270–273.

[4]Marvin J. Greenberg, *Euclidean and Non-Euclidean Geometries. Development and History* (Freeman, San Francisco, CA, 1980), 2nd ed., p. 187.

[5]Henri Poincaré described in *Science and Hypothesis* (Dover, New York, NY, 1952), pp. 70–71 (originally published in French, 1902), a three-dimensional version of this model:

> Suppose, for example, a world enclosed in a large sphere and subject to the following laws: the temperature is not uniform; it is greatest at the centre, and gradually decreases as we move towards the circumference of the sphere, where it is the absolute zero. The law of this temperature is as follows: if *R* is the radius of the sphere, and *r* the distance from the centre, then the absolute temperature will be proportional to R^2-r^2. Suppose that in this world the linear dilation of any body is proportional to its absolute temperature. A moving object will become smaller and smaller as it approaches the circumference of the sphere. Although from the point of view of our ordinary geometry this world is finite, to its inhabitants it will appear infinite. As they approach the surface of the sphere they will become colder and colder, and at the same time smaller and smaller. The steps they take are also smaller and smaller, so that they can never reach the boundary of the sphere. If to us geometry is only the study of the laws according to which rigid solids move, to these imaginary beings it will be the study of the laws of motions of solids deformed by the differences in temperature alluded to.

[6]In spite of the name, it was Beltrami fourteen years before Poincaré who first discovered this model. See Tristan Needham, *Visual Complex Analysis* (Clarendon, Oxford, 1997), p. 315.

[7]William P. Thurston, *Three-dimensional Geometry and Topology* (Princeton U.P., Princeton, NJ, 1997), Vol. 1, p. 66.

[8]If we consider complex coordinates on the disk, a Lorentz transformation corresponds to a Möbius transformation of the form: $z \mapsto (az+b)/(bz+\bar{a})$, where *a* and *b* are complex numbers determined by the Lorentz transformation and satisfying $|a|>|b|$, see, for example, Ref. 6, pp. 319–322.

[9]$\cosh|RT| = \cosh|RS|\cosh|ST| - \sinh|RS|\sinh|ST|\cos\alpha$, where α is the angle between the velocities $\mathbf{v}_{\mathcal{S}|\mathcal{R}}$ and $\mathbf{v}_{\mathcal{T}|\mathcal{S}}$, see, for example, Ref. 4, p. 337 or in Ref. 7, p. 81.

[10]The classical way to obtain the hyperbolic metric from the formula of composition of velocities proceeds as follows [see, for example, problems 1 and 2, Sec. 5 of L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, London, 1962)]: If $\mathbf{v}_1$, $\mathbf{v}_2$ are the velocities of two particles as measured in a given coordinate system and $\mathbf{v}_{\rm rel}$ is their relative velocity in this inertial frame, then $v_{\rm rel}^2 = [(\mathbf{v}_1-\mathbf{v}_2)^2 - (\mathbf{v}_1\times\mathbf{v}_2)^2]/(1-\mathbf{v}_1\cdot\mathbf{v}_2)^2$. The element of length dl_v in the relativistic ''space of velocities'' is the relative velocity of two particles with velocities $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$. Therefore, $dl_v^2 = [dv^2/(1-v^2)^2 + v^2/(1-v^2)](d\theta^2 + \sin^2\theta d\phi^2)$, where θ and ϕ are the polar angle and the azimuthal angle of $\mathbf{v}$. If we introduce the new variable χ by means of the identity $\mathbf{v} = \tanh\chi$, the element of length takes the form $dl_v^2 = d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)$. From the geometrical point of view, this is the element of length in the three-dimensional Lobachevski space, that is, the space of constant negative curvature.

[11]James W. Cannon, William J. Floyd, Richard Kenyon, and Walter R. Parry, ''Hyperbolic Geometry'' in *Flavors of Geometry*, edited by Silvio Levy (Cambridge U.P., Cambridge, 1997), pp. 59–116. Also in Ref. 7, p. 67.

[12]H. S. M. Coxeter, *Introduction to Geometry* (Wiley, New York, 1969), p. 299.

[13]See Ref. 4, p. 334.

[14]W. Rindler, *Essential Relativity* (Springer-Verlag, New York, 1986), 2nd ed., p. 47.

[15]Gauss measured the sum of the angles of a triangle formed by three peaks: Broken, Hohehagen, and Inselsberg. The triangle sides were 69, 85, and 197 km. He found that the sum exceeded 180° by 14″85. The experiment did not demonstrate anything since the experimental error was bigger than the excess, so that the correct sum could be 180° or even less. As Gauss realized, the triangle was small and, since the defect is proportional to the area, only a big triangle could reveal any significant difference of the sum of the angles from 180°, see, for example, Morris Kline, *Mathematics and the Physical World* (Dover, New York, 1989), pp. 443–463.

[16]Let α' be the angle in the frame $K_{\mathcal{B}}$ between the light signals coming from $\mathcal{R}_2$ and $\mathcal{R}_3$ and arriving at $\mathcal{R}_1$. It is easy to see that $\tan(\alpha'/2) = \sqrt{(1-v^2)}/(\sqrt{3}+2v)$. To get the angle $\alpha/2$ in the frame $K_{\mathcal{R}_1}$, we have to use the aberration formula (see p. 58 of Ref. 14): $\tan(\alpha/4) = [(1-v)/(1+v)]^{1/2}\tan(\alpha'/4)$. Thus a straightforward calculus yields the expected formula: $\tan(\alpha/2) = \sqrt{(1-v^2)}/\sqrt{3}$.

[17]See Ref. 14, pp. 51–53.

THERMODYNAMICS VERSUS INTUITION

Thus, Nernst believed that his paper of 1889 did make explicit use of thermodynamics. But although he used and accepted the validity of the thermodynamic derivation, and although he considered it adequate, yet he nonetheless hoped to provide added ''*Anschaulichkeit*,''—or ''intuition'' or ''physicality''—to the underlying processes.

Diane Kormos Barkan, *Walther Nernst and the Transition to Modern Physical Science* (Cambridge U.P., New York, 1999), p. 75.