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Maximal Modular Inner Ideals in Jordan Systems

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ABSTRACT

We give a description of maximal-primitizers and maximal-weak primitizers of Jordan systems, and prove two conjectures that had been previously raised: maximal modular inner ideals are maximal among all inner ideals, and maximal weakly modular inner ideals are modular.

Key Words: Jordan system; Primitive; Inner ideal.
INTRODUCTION

Zelmanov’s Prime Dichotomy Theorem (1983a) required the characterization of the Jacobson radical of a unital linear Jordan algebra as the intersection of the cores of all maximal inner ideals. This was later extended to (nonunital) quadratic Jordan algebras by Hogben and McCrimmon in (1981), where they defined a notion of modularity that parallels the associative definition of one-sided modular ideal, and proved that the Jacobson radical of a quadratic Jordan algebra is the intersection of all maximal-modular inner ideals (and of the cores of all maximal-modular inner ideals). The notion of modular inner ideal was generalized to Jordan pairs and triple systems in Zelmanov (1983b, 1985) together with the corresponding characterizations of the Jacobson radical, which again played a fundamental role in the structure theory.

In this paper we describe maximal-modular inner ideals in Jordan systems, extending work done in Anquela et al. (1993a, b) and Montaner (1993) for Jordan algebras to Jordan pairs and triple systems. In particular, we answer in the affirmative the following two open questions concerning maximal-modular inner ideals.

First, recall that a maximal modular right ideal of an associative algebra is always maximal among all right ideals. Hogben and McCrimmon (1981, p. 169) posed the question of the validity of the Jordan analogue of that fact:

Is every maximal-modular inner ideal always maximal among all inner ideals?

This was settled in Anquela et al. (1993a) for Jordan algebras. Here we will extend that result to pairs and triple systems.

A second question, raised by Anquela and Corteš (1998, 4.19), concerns the notion of weak modularity:

Are maximal-modularity and maximal-weak modularity equivalent notions for Jordan pairs and triple systems?

We recall that this question was answered in the affirmative for algebras in Hogben and McCrimmon (1981).

The argument leading to those results is a generalization of the one in Anquela et al. (1993a, b) and Montaner (1993), where the study of primitizers splits into two cases: PI-primitizers and non PI-primitizers. When dealing with pairs and triple systems, the workable notion of
polynomial identity seems to be that of homotope polynomial identity. Accordingly, our investigation of maximal inner ideals splits into the study of primitizers which satisfy a homotope-PI in the Jordan system (i.e., they satisfy a given PI in any homotope) and primitizers which do not.

After a first section of preliminaries where we recall basic definitions and known results, we first study weak primitizers of Jordan systems, proving that we can build a primitizer out of every weak primitizer, and that primitizer is PI in some particular homotope if the original weak primitizer was. This allows us to uniformize the proofs for weak primitizers and primitizers (and it also gives a non structural proof of a result of Anquela and Cortés, 1998 that establishes the equivalence between primitivity and weak primitivity). In Sec. 2 we consider (weak) primitizers in pairs with capacity, and solve the above conjectures in that case. Moreover, we show that weak modularity is the same as modularity in pairs with capacity.

In Sec. 3 we consider the homotope-PI/non homotope-PI dichotomy, and prove that pairs with maximal primitizers which are homotope-PI in the above sense are simple with finite capacity, so that the results of Sec. 2 apply to them. The rest of the paper is devoted to the study of primitizer which are not homotope-PI in the above sense. Sections 4 and 5 are technical, we prove in Sec. 4 that properness of inner ideals containing a maximal primitizer is the same as not containing a modulus of the primitizer, and in Sec. 5 we construct a particular class of polynomials which interacts well with inner ideals, simplifying a construction of Anquela and Cortés (1996). In Sec. 6 we apply the results of Secs. 4 and 5 to non homotope-PI maximal primitizers, showing that they are the intersection of a maximal modular right ideal of a *-envelope of the Jordan system with the Jordan system itself. That is used to solve the above conjectures in this case. Finally, we collect the results of the previous sections in Sec. 7, and solve the conjectures in the general case.

We remark that the homotope-PI/non homotope-PI dichotomy seems the right one to describe primitizers, not only in Jordan pairs and triple systems, but also in Jordan algebras. In fact, while having a PI maximal primitizer just implies that an algebra is simple and unital, having a maximal primitizer which is homotope-PI in the algebra gives finite capacity, and therefore the study of these algebras is subsumed in the well developed theory of algebras with principal dcc. On the other hand, primitizers that do not satisfy a PI in some homotope can be “lifted” to some associative envelope, which is the basic fact for non-PI primitizers (Anquela et al., 1993b, 3.3).
Throughout $\Phi$ will be a fixed unital commutative ring.

0.1. We will work with Jordan pairs, triple systems, and algebras over $\Phi$. We refer to Jacobson (1981), Loos (1975), McCrimmon and Zelmanov (1988) for notation, terminology, and basic results. We record in this section some of those notations and results.

A Jordan algebra $J$ has products $x^2$ and $U_{xy}$, quadratic in $x$ and linear in $y$, whose linearizations are $x \circ y = V_{xy} = (x + y)^2 - x^2 - y^2$, and $U_{x,y} = V_{x,y} = \{x, z, y\} = U_{x+z,y} - U_{x,y} - U_{z,y}$, respectively.

A Jordan Pair $V = (V^+, V^-)$ has products $Q_{xy}$ for $x \in V^+$ and $y \in V^-$, with linearizations $Q_{x,y} = D_{x,y} z = \{x, y, z\} = Q_{x+z,y} - Q_{x,y} - Q_{y,z}$. We also mention the important Bergmann operators given by $B_{x,y} z = z - D_{x,y} z + Q_{x,y} z$ for $x, z \in V^+$, $y \in V^-$, which satisfy $Q B_{x,y} z = B_{x,y} Q z$.

A Jordan triple system $T$ has products $P_{xy}$ whose linearizations are $P_{x,y} = L_{x,y} = \{x, z, y\} = P_{x+z,y} - P_{x,y} - P_{y,z}$. The Bergmann operator in triple systems is given by $B_{x,y} z = z - L_{x,y} z + P_{x,y} z$.

0.2. Doubling a Jordan triple system $T$ produces a Jordan pair $V(T) = (T, T)$ with $Q_{xy} = P_{xy}$. Reciprocally, each Jordan pair $V = (V^+, V^-)$ gives rise to a polarized triple system $T(V) = V^+ \oplus V^-$ with product $P_{x+y, y} = Q_{x,y} \oplus Q_{x,y}$, $y \in V^-$, $Q_{x,y} - Q_{x,y} - Q_{y,z}$. Niceness conditions such as nondegeneracy, primeness, strong primeness and others are inherited by the polarized triple system of a Jordan pair. However this does no longer hold in the reverse direction, from Jordan triple systems to their double Jordan pairs. To remedy that situation, D’Amour and McCrimmon (1995, p. 229), and Anquela and Cortés (1996, p. 667) defined tight doubles.
Given a Jordan triple system $T$, a tight double of $T$ is a quotient pair $V(T)/I = (T/I^+, T/I^-)$ where $I$ is an ideal $(I^+, I^-)$ of $V(T)$ which is maximal with respect to $I^+ \cap I^- = \{0\}$ (so that the $I^\sigma$ are semi-ideals of $T$, but they may not be ideals). These always exist and share niceness properties with $T$ (see 5.2 and 5.3 of Anquela and Cortés, 1996).

For a strongly prime $J$, the ideal $I$ is unique up to the exchange involution: if $V(J)/L$ is another tight double, then either $L = I$ or $L^{op} = (L^-, L^+) = I$. Indeed, if $L$ is any ideal of $V(J)$ with $L^+ \cap L^- = \{0\}$, setting $N = L + I$ and $M = L^{op} + I$, the sets $N^+ \cap N^-$ and $M^+ \cap M^-$ are ideals of $J$, and they are orthogonal: $P_N \cap P_{M^+} \cap P_{M^+} \subseteq (P_{L^+} \cap L^+) \cap (P_{L^+} \cap L^-) = I^+ \cap I^- = \{0\}$, hence either $N^+ \cap N^- = \{0\}$ or $M^+ \cap M^- = \{0\}$ by strong primeness of $J$, which implies that either $I = N$ or $I = M$ by maximality of $I$.

0.3. We refer to D’Amour (1992), Anquela and Cortés (1996) for notations and basic results on associative pairs and triple systems.

An associative pair $R = (R^+, R^-)$ gives rise to a Jordan pair $R^{(\sigma)}$ with products $Q_{a,b}x^{\sigma} = x^\sigma y^{\sigma}x^\sigma$, $\sigma = \pm$. Similarly, an associative triple system $R$ gives rise to a Jordan triple system $R^{(\sigma)}$ with product $P_{x,y} = x^{\sigma}y^{\sigma}$. A Jordan system is special if it is a subsystem of some $R^{(\sigma)}$ for an associative system $R$. Among those, important examples are ample subsystems $H_0(R, *)$ of associative systems $R$ with involution $*$, subspaces $H_0 \subseteq H(R, *)$ which contain all norms $aa^*$ and traces $a + a^*$, $a \in A$, and all products $aha^*$ for $a \in A$ and $h \in H_0$.

Since it will be important for us later, we recall the notion of modular left or right ideal of an associative system. If $R$ is a Jordan pair, $L \subseteq R^\sigma$ is a left ideal, and $(a, b) \in R^\sigma \times R^{-\sigma}$, $L$ is said to be $(a, b)$-modular if, for any $x \in R^\sigma$, $x - abx \in L$. Modular right-ideals of Jordan pairs are defined symmetrically, and modular left or right ideals of associative triple systems are modular right or left ideals of the associative pair $V(R) = (R, R)$.

0.4. Let $(V^+, V^-)$ be a Jordan pair and $a \in I^\sigma$, where $\sigma = \pm$. The $a$-homotope $V^\sigma$, denoted by $(V^{-\sigma})^a$, is the Jordan algebra over the $\Phi$-module $V^{-\sigma}$ with operations $U_{(a)}y^{\sigma} = Q_{\sigma}Q_{xy^{\sigma}}$ (linearized to $\{x^\sigma, y^\sigma, z^\sigma\} = \{x^\sigma, Q_{\sigma}y^{\sigma}, z^\sigma\}$, and $(x^{-\sigma})^2 = Q_{x^\sigma}a$ (linearized to $x^{-\sigma}Q_{(a)}y^{-\sigma} = \{x^\sigma, a, y^\sigma\}$).

The set $\text{Ker } a$ of all $x^{-\sigma} \in V^{-\sigma}$ such that $Q_{ax}x^{-\sigma} = Q_{ax}Q_{x^-}a = 0$ (or simply $Q_{ax}x^{-\sigma} = 0$ if $V$ is nondegenerate) is an ideal of $a$, so that the quotient $V^{-\sigma}/(V^{-\sigma})^a/\text{Ker } a$ is again a Jordan algebra. This is called the local algebra of $V$ at $a$. 


For triple systems and Jordan algebras, homotopes and local algebras are defined in the same way: just delete the superscripts $s$ from the previous definitions. We refer to D’Amour and McCrimmon (1995) for a thorough study of local algebras.

Local algebras of Jordan pairs can be viewed through the theory of subquotients as developed by Loos and Neher (1994): If $V = (V^+, V^-)$ is a Jordan pair and $M \subseteq V^\sigma$ is an inner ideal of $V$, the subquotient of $V$ determined by $M$ is the pair $S$ given by $S^\sigma = M$ and $S^{-\sigma} = V^{-\sigma}/\text{Ker}_V M$, $M$, where $\text{Ker}_V M$ (or simply $\text{Ker} M$ if there is not ambiguity) is the set of $x \in V^{-\sigma}$ for which $QMx = QMxM = 0$. (Again, the second condition is superfluous if $V$ is nondegenerate.)

When $M = \Phi a + Q_a V^{-\sigma}$ is the principal ideal determined by $a \in V^\sigma$, the subquotient $S$ determined by $M$ in $V$ has $S^{-\sigma} = V_a^{-\sigma}$, and $S$ is isomorphic to the double $(V_a^{-\sigma}, V_a^{-\sigma})$ (see Montaner, 1999, 0.4). Moreover, if $a \in 1^\sigma$ is regular, we can complete it to an idempotent $e = (e^+, e^-)$ with $e^\sigma = a$, and the subquotient determined by $M$ is (isomorphic to) the Peirce space $V_2(e)$ (Loos and Neher, 1994, 1.12).

0.5. The Jordan algebra version of the notion of primitivity mimics the intrinsic characterization of primitivity in associative algebras by means of modular ideals, so it requires a Jordan version of modularity. Recall (Hogben and McCrimmon, 1981) that an inner ideal $K$ of a Jordan algebra $J$ is called $e$-modular, for an element $e \in J$ which is called a modulus for $K$, if it satisfies:

(i) $U_{1-e} J \subseteq K$.
(ii) $\{K, J, 1 - e\} \subseteq K$.
(iii) $(1 - e) \odot K \subseteq K$.
(iv) $e - e^2 \in K$.

This is equivalent to the fact that $\hat{K} = K + \Phi(1 - e)$ be an inner ideal of the unital hull $\hat{J}$ of $J$. Moreover, an inner ideal of a unital algebra $J$ is $e$-modular if and only if it contains $1 - e$ (Hogben and McCrimmon, 1981, 2.9).

An inner ideal $K$ of a Jordan algebra $J$ is called weakly $e$-modular if it satisfies (i) and (ii) (see Hogben and McCrimmon, 1981, 2.8). The element $e$ is then called a weak modulus for $K$.

0.6. Modularity of inner ideals in Jordan pairs and triple systems is defined as modularity in some homotope (see Hogben and McCrimmon, 1981). Let $V = (V^+, V^-)$ be a Jordan pair, and $(a^+, a^-) \in V^+ \times V^-$. 
An inner ideal $K \subseteq V^n$, $\sigma = \pm$, is called $a^\sigma$-modular at $a^{-\sigma}$, or $(a^\sigma, a^{-\sigma})$-modular, and $(a^\sigma, a^{-\sigma})$ is called a modulus for $K$, if $K$ is an $a^\sigma$-modular inner ideal of the homotope $V^{\sigma(a^{-\sigma})}$, i.e., if it satisfies:

(i) $B_{a^\sigma, a^{-\sigma}}V^n \subseteq K$,
(ii) $\{k, a^{-\sigma}, x\} = k, Q_{a^{-\sigma}x}, a^\sigma\} \subseteq K$, for all $x \in V^n, k \in K$,
(iii) $\{K, a^{-\sigma}, a^\sigma\} \subseteq K$,
(iv) $a^\sigma - Q_{a^{-\sigma}a^{-\sigma}} \in K$.

The inner ideal $K$ is said to be weakly $(a^\sigma, a^{-\sigma})$-modular if it satisfies (i) and (ii), i.e., if it is weakly $a^\sigma$-modular in the homotope $V^{\sigma(a^{-\sigma})}$ (Anquela and Cortés, 1998). In this case $(a^\sigma, a^{-\sigma})$ is called a weak modulus for $K$.

Modularity and weak modularity of Jordan triple systems are defined in the same way, so that an inner ideal $K$ of $J$ is (weakly) $(a, b)$-modular, where $a, b \in J$, if it is (weakly) $a$-modular in the homotope $J^{(b)}$. In this case $(a, b)$ is called a (weak) modulus for $K$.

0.7. Having many moduli at our disposal is important to check properness: a (weakly) $(a, b)$-modular inner ideal $K$ will be proper if and only if $a \not\in K$. (If $a \in K$, for any $x \in J$, $\{a, b, x\} = \{a, Pbx, a\}$ (by 0.6(ii)) $\equiv 0$ (mod $K$) since $a \in K$, hence $x = B_{a,b}x + \{a, b, x\} - P_aP_{bx} \in K$, by 0.6(i)), $a \in K$, and $\{a, b, x\} \subseteq K$.)

0.8. We can get new moduli from known ones (Hogben and McCrimmon, 1981, 2.10; D’Amour and McCrimmon, 1995, 5.13):

(1) If $K$ is an inner ideal of the Jordan algebra $J$, and $e$ is a modulus for $J$, then $e^\sigma$ and $e + k$ are moduli of $K$ for any $n \geq 1$, $k \in K$.

(2) If $K$ is an inner ideal of the Jordan pair or triple system $J$, and $(a, b)$ is a modulus for $K$, then $(a^{(m,n)}, b^{(m,n)})$, $(a + k, b)$ and $(a, b + P_bk)$ are moduli of $K$ for any $n, m \geq 1$ and any $k \in K$.

The fact that the last two pairs of elements are moduli is a particular case of a more general result. For an $(a, b)$-modular inner ideal $K$ of a Jordan system $J$, define the $a$-dual of $K$ as the set $K^{(a)}$ of $x \in J$ such that $P_{ax} \in K, P_aP_J \subseteq K, \{K, x, a\} \subseteq K$ and $\{K, P_J, a\} \subseteq K$. Then $K^{(a)}$ is a $(b, a)$-modular inner ideal of $J$ containing $P_bK$, and one has Montaner (2002, Proposition 3.4):

(3) If $K$ is an inner ideal of a Jordan system $J$, and $(a, b)$ is a modulus for $J$, then $K$ is $(a + k, b + x)$-modular for any $k \in K$ and any $x \in K^{(a)}$. 
The operations of (1) and (2) also work for weak moduli:

(4) If $K$ is an inner ideal of the Jordan algebra $J$, and $e$ is a weak modulus for $J$, then $e^n$ and $e+k$ are weak moduli for $J$ for all $n \geq 1$, $k \in K$.

Proof. That $e+k$ is again a modulus is straightforward. Next, note that the equality $\{1-e, x, e-e^{n+1}\} = U_{1-e}x + (e-e^n) + e^n$) for all $n \geq 1$ and all $x \in J$ follows from Macdonald’s Theorem, whence $\{1-e, x, e-e^{n+1}\} \subseteq K$ for all $n \geq 0$.

Then we have $U_{1-e}x = U_{1-e}x + (e-e^n) + e^n \equiv U_{1-e}x + U_{e-e^n}x \equiv U_{e-e^n}Kx \equiv Kx$ (mod $K$). Hence, inductively, $U_{1-e}Kx \subseteq K$ for all $n \geq 1$.

On the other hand, in a unital hull $J$ of $J$, we have the equality $U_{1-e}x = e^n - 2e^{n+1} + e^{n+2}$. This gives $1 - e^{n+2} = 2(1-e^{n+1}) - (1-e^n) = U_{1-e}x$ for all $n \geq 0$, hence, for all $k \in K$ and $x \in J$, $\{1-e^n, x, k\} \equiv 2\{1-e^n, x, k\}$ (mod $K$). Thus, inductively, $\{1-e^n, J, K\} \subseteq K$ for all $n \geq 1$.

(5) If $K$ is an inner ideal of the Jordan system $J$, and $(a, b)$ is a weak modulus for $K$, then $(a^{n, b}, b^{m, a})$, $(a+k, b)$, and $(a+b+Pp, k)$ are weak moduli of $K$ for all $k \in K$, and all $n, m \geq 1$.

Proof. We write $x \equiv y$ for $x - y \in K$. That $a+b+Pp$ is a weak modulus is straightforward. On the other hand, if $k \in K$ and $x \in J$, $B_{a, b+Pp, k}x = B_{a, b}x + PpPp_{a}x - \{a, Pp_{k}, x\} + Pp_{a}x$, $Pp_{b}$. Now, we have $\{a, Pp_{k}, x\} = \{a, Pp_{k}, x\} - \{Pp_{a}, x\} = \{a, Pp_{k}, x\} - \{Pp_{a}, x\} + \{a, Pp_{k}, x\}$ (by JP7) $\equiv \{a, k, b, x\} - \{a, b, x\} + \{a, k, b, x\} - \{a, b, x\} = \{a, k, b, x\}$. Also, $Pp_{a}Pp_{b}x = Pp_{a}Pp_{b}x = Pp_{a}Pp_{b}x - Pp_{a}Pp_{b}x + \{a, b, Pp_{b}x\} \equiv \{a, b, Pp_{b}x\}$ (by weak modularity and innerness of $K$) $\equiv \{a, b, k, Pp_{b}x, x\} - \{a, b, k, Pp_{b}x, x\}$ (by JP8) $\equiv \{a, Pp_{b}, k, b, x\} - \{a, Pp_{b}, k, b, x\}$ (by JP1) $\equiv \{k, b, x\} - \{a, Pp_{b}, k, b, x\}$ (by weak modularity) $\equiv \{k, b, x\} - \{a, Pp_{b}, k, b, x\}$ (by JP9) $\equiv \{x, b, Pp_{b}\} - \{a, b, Pp_{b}\}$ (mod $K$) $\in K$ (by weak modularity). Therefore $B_{a, b+Pp, k}x \subseteq K$. Now, if $k', Pp_{a}x$, $a' = \{k', Pp_{k}, x\} - \{k', Pp_{a}, x\} + \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\}$ (by identities JP7, JP1) $\equiv \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\}$ (by weak modularity) $\equiv \{k', Pp_{a}, x\} - \{k', Pp_{a}, x\}$ (by weak modularity) $\in K$. Therefore $(a, Pp_{b})$ is a weak modulus for $K$. 


Next, since weak $\times$-modularity at $b$ is the same as weak $\times$-modularity in the homotope $J^{(a)}$, (1) implies that $(a^{(m,a)}b)$ is a weak modulus for any $m \geq 1$. Finally, to see that $(a^{(m,a)}b)$ is a weak modulus for $m \geq 2$, we first note $p_{(m,a)}^{(m-1,b)}$, and compute in the $b$-homotope $B_{x^{(m,a)}b^{(m,a)}} = x - \{d^m, d^{m-1}x\} + U_{x^{(m,a)}b^{(m,a)}} = x - d^{m+1} \circ x + U_{x^{(m,a)}b^{(m,a)}}$. This has been proved in (Hogben and McCrimmon, 1981, 5.5; Zelmanov, 1983b) that weak primitivity is the same as weak primitivity, and we will obtain that result as a corollary of the constructions of Sec. 1.

0.9. Primitive Jordan systems (algebras, triple systems or pairs) (D’Amour, 1995; Hogben and McCrimmon, 1981; Zelmanov, 1983b) are defined as systems $J$ having a primitizer, a proper modular inner ideal $K$ which complements nonzero ideals: $K \oplus I = J$ for all nonzero $I \lhd J$ if $J$ is an algebra or a triple system, and $K \oplus I' = V'$ for all nonzero $I = (I^+, I^-) \lhd J$, and $K \subset V'$, if $J = (V^+, V^-)$ is a Jordan pair. The definition of primitivity for pairs splits into two notions, (+)-primitivity and (−)-primitivity, according to whether the primitizer $K$ is contained in $V^+$ or $V^-$, respectively. However, it has been proved in Anquela and Cortés (1998) (see also Montaner, 2002) that these are equivalent notions.

Similarly, one defines weakly primitive Jordan systems as systems having a weak primitizer by taking weakly modular inner ideals instead of modular inner ideals in the previous definition. It has been proved in Anquela and Cortés (1998, 4.17) that weak primitivity is the same as primitivity, and we will obtain that result as a corollary of the constructions of Sec. 1.

Primitivity is a strong regularity condition, since we have:

0.10. A (weakly) primitive Jordan system is strongly prime.

Proof. This has been proved in (Hogben and McCrimmon, 1981, 5.5; Anquela and Cortés, 1996, 3.7; Anquela and Cortés, 1998, 4.9). Note, however, that the proof for weak primitive systems makes use of the long computations of Hogben and McCrimmon (1981). This can be avoided in the following way.

Take a Jordan triple system $J$ with weak primitizer $K$ with weak modulus $(a, b)$. That $J$ is nondegenerate follows as in Anquela and Cortés...
(1996, 3.7) using 0.8(5). Next, if \( I \) and \( L \) are nonzero orthogonal ideals of \( J \), \( P_I L = 0 \), then \( J = K + I \) since \( K \) is a weak primitizer, hence there is \( c \in I \) with \( a - c \in K \). Thus \( (c, b) \) is a weak modulus for \( K \) by 0.8(5). Similarly, \( J = K + L \), hence \( (2, b) \in P_J b = P_{K + L} b \subset P_K b + L \) thus there is \( k \in P_K b \) and \( x \in L \) with \( c(2, b) = x + k \). Then \( (4, b) = P_j P_k (2, b) = P_j P_k k + P_j P_k x = P_j P_k y \) (since \( P_j P_k x \in P_I L = 0 \)) = \( B_0 k - k + \{ c, b, k \} \equiv \{ c, b, k \} \) (mod \( K \)). Now, \( k \) is a sum of elements of the form \( P_k b \), and we have \( \{ c, b, P_k b \} = \{ c, P_i k', k' \} \) (by JP2) \( \equiv \{ k', b, k' \} \) (mod \( K \), by weak modularity) \( \in K \), hence \( \{ c, b, k \} \in K \), and \( c(4, b) \in K \), but this contradicts properness of \( K \) by 0.7 since \( (4, b) \) is a weak modulus for \( K \) by 0.8(5).

The argument is the same for pairs and algebras (formally allowing \( b = 1 \)).

0.11. The fact that primitizers \( K \) of a Jordan system \( J \) complement nonzero ideals implies that they are core-less: \( \text{Core}(K) = 0 \), where \( \text{Core}(K) \) is defined as the sum of all ideals contained in \( K \) if \( J \) is an algebra or a triple system, and \( \text{Core}(K) \) is the sum of all nonzero ideals \( I = (I^+, I^-) \) with \( I^+ \subseteq K \) if \( J = (V^+, V^-) \) is a Jordan pair, and \( K \subseteq V^0 \) for \( \sigma = + \) or \(-\).

Since, for a nonzero ideal \( I \), the inner ideal \( K + I \) (resp. \( K + I' \) in the pair case) is again modular with the same modulus as \( K \), the properness criterion 0.7 implies that a core-less \( K \) will be a primitizer if it is maximal among proper modular inner ideals with the given modulus.

A (weakly) modular inner ideal is called maximal-(weakly) modular if, for some of its (weak) moduli \( (a, b) \) (\( e \), in the algebra case), it is maximal among all proper (weakly) \( (a, b) \)-modular ((weakly) \( e \)-modular, respectively) inner ideals. By an easy argument using Zorn’s Lemma, it can be shown that any proper (weakly) modular inner ideal \( K \) is contained in a proper maximal-(weakly) modular inner ideal \( M \), and if \( K \) complements nonzero ideals, \( M \) is core-less. Therefore, primitivity is equivalent to the existence of a core-less maximal-modular inner ideal.

Maximal modular inner ideals are important for the characterization of the Jacobson radical through modular inner ideals (Hogben and McCrimmon, 1981, 5.6; Zelmanov, 1983b, Lemma 6; Zelmanov, 1985, Theorem 8; Montaner, 2002, Theorem 1.7).

0.12. Modularity in Jordan pairs and triple systems can be related by means of the functors \( T \) and \( V \). The following assertions are proved
in Montaner (2002, 1.6) for modular inner ideals, and the same proofs work for weakly modular inner ideals.

A Jordan triple system $J$ gives rise to the Jordan pair $V(J)$. If $\sigma = \pm$, $S \subseteq J$ and $x \in J$, we write $S^\sigma$ for $S$ as a subset of $V(J)^\sigma$, and $x^\sigma$ for $x$ viewed as an element of $V(J)^\sigma$. We have the obvious properties:

Let $J$ be a Jordan triple system, $K$ be an inner ideal of $J$, $a, b \in J$, and $\sigma = \pm$. Then:

(a) $K$ is (weakly) $(a, b)$-modular if and only if $K^\sigma$ is (weakly) $(a^\sigma, b^\sigma)$-modular.

(b) $K$ is maximal-weakly modular for the modulus $(a, b)$ if and only if $K^\sigma$ is maximal-weakly modular for the modulus $(a^\sigma, b^\sigma)$.

We note that $K$ being core-less or complementing nonzero ideals does not imply the same properties for $K^\sigma$, and thus $K^\sigma$ need not be a primitizer if $K$ is (see D’Amour and McCrimmon, 1995, 5.6). However we have:

(c) If $K$ is a maximal (weak) primitizer of $J$, $\sigma = \pm$, then $V(J)/\text{Core}(K^\sigma)$ is a tight double of $J$ with maximal (weak) primitizer $K^\sigma/\text{Core}(K^\sigma)^\sigma$.

Proof. Clearly $K^\sigma/\text{Core}(K^\sigma)^\sigma$ is a maximal primitizer of $V(J)/\text{Core}(K^\sigma)$ by (b), so it remains to see that $V(J)/\text{Core}(K^\sigma)$ is a tight double of $J$.

Denote $\text{Core}(K^\sigma) = I = (I^+, I^-)$. Then $I^+ \cap I^- = \emptyset$. Now, if $L = (L^+, L^-)$ is an ideal of $V(J)$ properly containing $I$ with $L^+ \cap L^- = \emptyset$, by definition of $\text{Core}(K^\sigma)$, $L^\sigma \subseteq K^\sigma$, hence $K + L^\sigma$ is easily seen to be a (weakly) modular inner ideal for the modulus $(a, b)$ of $K$, properly containing $K$. Thus $K + L^\sigma = J$. On the other hand, if $L^- \subseteq K$, then $L^\sigma = (L^-, L^+)$ has $\text{Core}(K^\sigma) = I$, and $I^+ \subseteq L^{-\sigma} \subseteq L^\sigma \subseteq K$ implies $I^+ + I^- \subseteq K$, but this is an ideal of $J$ and $K$ is core-less, hence $I^+ + I^- = 0$, and $L^- = 0$. But then $L^+ + L^- = L^\sigma$ is a nonzero ideal of $J$ (since it properly contains $I^+ + I^- = 0$), and $P_JL = P_JL^\sigma \subseteq L^- = 0$, contradicting strong primeness of $J$ (0.10). Thus $L^- \subseteq K$, and as before $K + L^{-\sigma} = J$. Thus there are $x^\delta \in L^\delta$ and $k^\delta \in K$, $\delta = \pm$, such that $a = x^\delta + k^\delta$. Then $(x^\delta, b)$ is a (weak) modulus for $K, \delta = \pm$, by 0.8, and $x^+ = B_{x^-, b}x^+ + \{x^-, b, x^+\} - P_xP_xB_{x^+, x^+} = B_{x^+, x^+}$ (since $\{x^-, b, x^+\} - P_xP_xB_{x^+, x^+} \in L^+ \cap L^- = 0$) in $K$, contradicting properness of $K$ by 0.7.

Consider now a Jordan pair $V = (V^+, V^-)$. This gives rise to a Jordan triple system $T(V) = V^+ \oplus V^-$. We denote by $\pi^\sigma$ the projection
of $T(V)$ onto $V^\sigma$, $\sigma = \pm$, and identify $V^\sigma \subseteq T(V)$. Also, if $S = (S^+, S^-)$ is a pair submodule of $V$, we write $T(S) = S^+ \oplus S^- \subseteq T(V)$.

Let $V$ be a Jordan pair, $\sigma = \pm$, $K \subseteq V^\sigma$ be an inner ideal, $a \in V^\sigma$ and $b \in V^{-\sigma}$. Write $K = K \oplus V^{-\sigma} \subseteq T(V)$. Then:

(d) For any $a', b'$ with $\pi^+(a') = a$ and $\pi^-(b') = b$, $K$ is (weakly) $(a, b)$-modular if and only if $\hat{K}$ is (weakly) $(a', b')$-modular.

(e) $T(\text{Core}(K)) = \text{Core}(\hat{K})$, hence $K$ is core-less if and only if $\hat{K}$ is core-less.

(f) The inner ideal $\hat{K}$ is maximal-(weakly) modular for the modulus $(a, b)$ if and only if $K$ is maximal-(weakly)-modular for the modulus $(a, b)$.

(g) $K$ is a (weak) primitizer if and only if $\hat{K}$ is a (weak) primitizer.

These results allow the lifting of properties from $V$ to $T(V)$. In the reverse way we have:

Let $V$ be a Jordan pair, $K$ be an inner ideal of $T(V)$, and $a = a^+ \oplus a^-$, $b = b^+ \oplus b^- \in T(V)$. Denote $K^\sigma = \pi^\sigma(K)$, $\sigma = \pm$, and $\hat{K} = K^+ \oplus K^-$, the polarization of $K$. Then:

(h) $K$ is (weakly) $(a, b)$-modular if and only if $\hat{K}$ is (weakly) $(a, b)$-modular, and, in this case, $K$ is proper if and only if $\hat{K}$ is proper.

(i) $K$ is (weakly) $(a, b)$-modular if and only if $K^\sigma$ is (weakly) $(a^\sigma, b^{-\sigma})$-modular for $\sigma = +$ and $-$.

(j) $K$ is maximal-(weakly) modular for the modulus $(a, b)$ if and only if $\hat{K}$ is polarized, and, for some $\sigma = \pm$, $K^\sigma$ is maximal-(weakly) modular for the modulus $(a^\sigma, b^{-\sigma})$, and $K^{-\sigma} = V^{-\sigma}$.

Unlike for Jordan algebras, primitivity of Jordan triple systems and pairs is defined at a particular element (although it has been proved in Anquela and Cortés, 1998 that primitivity can be moved to any other element). It is then natural to study primitivity of Jordan systems through their local algebras (see D’Amour and McCrimmon, 1995; Anquela and Cortés, 1996, 1998). To do that, the key fact is that primitivity flows from pairs to their local algebras and back (Anquela and Cortés, 1996). We will only make use of one direction, which we record in the following global-to-local inheritance theorem (Lemma 9 of Zelmanov, 1983b and Theorem 6.1 of D’Amour and McCrimmon, 1995):
Theorem 0.13. If a Jordan pair \( V \) is primitive at \( b \in V^\sigma \) with primitizer \( K \subset V^\sigma \), then the local algebra \( V/b \) is primitive with primitizer \( K + \ker b/\ker b \).

This result stems from an important combinatorial result which we mention here for later use. Recall that if \( I \) is an ideal of a Jordan triple system \( J \), \( S \) is a \( \Phi \)-submodule of \( J \) and \( b \in J \), \( I \) is \( b \)-nil modulo \( S \) if for any \( x \in I \) there is \( n \geq 1 \) such that \( x^{(n,b)} \in S \). If \( V = (V^+, V^-) \) is a Jordan pair, \( S \subset V^\sigma \) is a submodule, and \( b \in V^\sigma \), an ideal \( I \) of \( V \) is \( b \)-nil modulo \( S \) if for any \( x \in I^\sigma \) there is \( n \geq 1 \) such that \( x^{(n,b)} \in S \).

Proposition 0.14. Let \( J \) be a Jordan pair or triple system and let \( b \in J \). If \( L \) is an ideal of \( J \) properly containing \( \ker b \), there exists a nonzero ideal \( I \) of \( J \) which is \( b \)-nil modulo \( L \).

Proof. See D’Amour and McCrimmon (1995, Secs. 3 and 4) for Jordan triple systems. The result from pairs readily follows from that case (see Anquela and Cortés, 1998, Theorem 4.10).

0.15. After Zelmanov’s theorems on classification of Jordan systems, one of the main features of Jordan theory is the use of hermitian polynomials. We now recall some of their properties.

Let \( SJT[X] \) be the free special Jordan triple system, and \( AT[X] \) be the free associative triple system on the infinite set of variables \( X \). The system \( AT[X] \) can be given a unique involution \( * \) that fixes the elements of \( X \). Then \( SJT[X] \) naturally embeds into the set of symmetric elements \( H(AT[X], *) \) which is, in turn, a special triple system. In general, these triple systems do not coincide, \( H(AT[X], *) \) contains for all odd \( n \) the \( n \)-tads, the associative polynomials:

\[
\{ x_1, \ldots, x_n \} = x_1 \cdots x_n + x_n \cdots x_1,
\]

which are not Jordan polynomials for \( n \geq 5 \).

(Recall that in an associative triple system \( R \) with involution \( * \), the trace of \( x \in R \) is \( \{ r \} = r + r^* \), hence the \( n \)-tads are the traces of the monomials of \( AT[X] \).)

A very important fact is the existence of nonzero hermitian ideals, \( I \)-ideals \( \mathcal{H}(X) \) of \( STJ[X] \) with are \( n \)-tad closed for all odd \( n \):

\[
\{ \mathcal{H}(X), \ldots, \mathcal{H}(X) \} \subseteq \mathcal{H}(X).
\]
Hermitian ideals are constructed by D'Amour in (1991) from the more general hearty $n$-tad eater ideals $H_n(X)$ (for odd $n$), which eat adic $m$-tads from any position, for any odd $m \leq n$ (see D'Amour, 1991).

Eater ideals were first defined for algebras (and, in fact, the construction of eater ideals for Jordan triple systems makes use of their existence for Jordan algebras). Here, one considers the free special Jordan algebra $SJ[x]$, which is naturally embedded in the free associative algebra $Ass[x]$, and defines hearty $n$-tad eater ideals $H_n(X)$ as before but allowing even values of $n$ (McCrimmon and Zelmanov, 1988).

0.16. We finally mention some facts from Jordan PI-theory. Recall that a polynomial $f(x_1, \ldots, x_n) \in FJ[x]$, the free Jordan algebra on the set $X$, is called essential if its image in the free special Jordan algebra $SJ[x]$ under the natural homomorphism has a monic leading term (as an associative polynomial). A Jordan PI-algebra is a Jordan algebra which satisfies some essential $f(x_1, \ldots, x_n)$. From Anquela et al. (1995, 1.1 and 5.2) together with Corollary to Theorem 3 of Loos (1991b), analogues of Kaplansky's Theorem and Posner's Theorem follow:

**Theorem 0.17.** Let $J$ be a Jordan PI-algebra. If $J$ is primitive then it is simple with finite capacity. If $J$ is strongly prime, then the central closure $\Gamma^{-1}J$ is simple with finite capacity.

0.18. The operant notion of Jordan PI-triple system or pair, is that of homotope-PI triple system or pair. We will use the notations of D’Amour and McCrimmon (1995) and Anquela and Cortés (1996). In particular, if $f(x_1, \ldots, x_n)$ is a polynomial in the free Jordan algebra $FJ[x]$ on a countable set of generators $X$, and $z$ is an element of the free Jordan triple system $FJT[x]$, the polynomial

$$f(z; x_1, \ldots, x_n) = f^{(z)}(x_1, \ldots, x_n)$$

is the image of $f$ under the only homomorphism $FJ[x] \rightarrow FJT[x]$ extending the identity on $X$. If $(X) \subseteq FJ(X)$, and $Y \subseteq FJ[X]$, we denote by $T(Y; X)$ the subset of $FJT[X]$ formed by the polynomials $f(y; x_1, \ldots, x_n)$ for $f(x_1, \ldots, x_n) \in T(X)$ and $y \in Y$.

A Jordan triple system $T$ satisfies a homotope polynomial identity (homotope-PI, for short) if there is a polynomial $f(x_1, \ldots, x_n)$ in $FJ[X]$ whose image in the free special Jordan algebra $SJ[x]$ has a monic term of highest degree (as an associative polynomial) and such that the
polynomial $f(y; x_1, \ldots, x_n)$ with $y \in X$ different from the $x_i$, vanishes under all substitutions of elements $y, x_i \in T$.

That definition extends to Jordan pairs $V$ by considering their associated triple system $T(V)$. Notice that, since for all $a^+ \oplus a^- \in T(V)$ the homotope $T(V)^{(a^+ \oplus a^-)}$ is isomorphic to the product $V^{a^+} \times V^{a^-}$, a polynomial $f(x_1, \ldots, x_n) \in \text{FJP}[X]$ is an identity of all homotopes of $T(V)$ if and only if it is an identity of all homotopes of $V$. We can rephrase it in the following way. Choose disjoint sets $X^+$ and $X^-$, and bijections $X \to X^+, x \mapsto x^\sigma, \sigma = \pm$, and consider the free Jordan Pair $\text{FJP}[X^+ , X^-]$ (see Neher, 1999). For any $y^\sigma \in X^{-\sigma}$, there is a homomorphism $\psi_{y^\sigma} : \text{FJP}[X] \to \text{FJP}[X^+, X^-]^{(y^\sigma)}$ induced by the bijection $X \to X^\sigma$. We denote the image of a polynomial $h = h(x_1, \ldots, x_n) \in \text{FJP}[X]$ by $\psi_{y^\sigma}(h) = h(y^{-\sigma}; x_1^\sigma, \ldots, x_n^\sigma)$. Now if $V$ and $f$ are as before, setting $f^\sigma = f(y^{-\sigma}; x_1^\sigma, \ldots, x_n^\sigma) \in \text{FJP}[X^+, X^-]^{y}$ for $\sigma = \pm$, where $y^\sigma \in X^\sigma$ and $y^\sigma \neq x_i^\sigma$, $f(y; x_1, \ldots, x_n)$ is an identity of $T(V)$ if and only if $(f^+, f^-)$ is an identity of $V$.

A question that naturally arises is the following. Let $V$ be a Jordan Pair, and $g, h \in \text{FJ}[X]$ be essential polynomials. Suppose that $(\psi_{y^-}(g), \psi_{y^+}(h))$ is an identity of $V$. Is it true that $V$ satisfies a homotope-PI? To prove that the answer is positive, we need first the following computation.

**Lemma 0.19.** Let $f(x_1, \ldots, x_n)$ be an element of $\text{FJ}[X]$, and let $y, z \in X, y, z \neq x_1, \ldots, x_n$. Write $g(x_1, \ldots, x_n, y) \in \text{FJ}[X]$ for the image of $f(y; x_1, \ldots, x_n) \in \text{FJT}[X]$ under the mapping $\text{FJT}[X] \to \text{FJ}[X]$ induced by the identity on $X$. Then

$$g(z; x_1, \ldots, x_n, y) = f(P_2y; x_1, \ldots, x_n)$$

and

$$P_2g(z; x_1, \ldots, x_n, y) = f(y; P_2x_1, \ldots, P_2x_n).$$

**Proof.** The second equality follows from the first and Lemma 2.1 of Anquela and Cortés (1996). For the first one, for any $t \in \text{FJT}[X]$, denote by $\phi_t : \text{FJ}[X] \to \text{FJT}[X]^{(t)}$ the only homomorphism of algebras which is the identity on $X$, and for any $t \in \text{FJ}[X]$, denote by $\psi_t : \text{FJ}[X] \to \text{FJT}[X]^{(t)}$ the only homomorphism of algebras which is the identity on $X$. Since $\phi_t(y) = y, \phi_t$ induces a homomorphism $\phi : \text{FJ}^{(t)} \to \text{FJT}[X]^{(t)}$ where $\phi(y) = \phi_t(y)$. Now, $\phi\psi_t : \text{FJ}[X] \to \text{FJT}[X]^{(t)}$ is the identity on $X$, and
therefore $\phi_{y} = \phi_{P_{y}P_{y}}$. Thus
\[ g(z; x_{1}, \ldots, x_{n}, y) = \phi_{y}f(x_{1}, \ldots, x_{n}) = \phi_{P_{y}P_{y}}(f(x_{1}, \ldots, x_{n})) = f(P_{2}y; x_{1}, \ldots, x_{n}). \]

Lemma 0.20. Let $V$ be a Jordan Pair, let $f \in FJ[X]$ be an essential polynomial, and fix $\sigma = +$ or $-$. If every homotope $V^{\sigma}(a)$ with $a \in V^{-\sigma}$ satisfies $f = 0$, then $V$ is homotope-PI.

Proof. Write $f = f(x_{1}, \ldots, x_{n})$ and take a variable $y \in X$, $y \neq x_{1}, \ldots, x_{n}$. Let $g = g(x_{1}, \ldots, x_{n}, y)$ be the image of $f$ in the algebra $FJ[X]$ (so that $g$ is the image of $f(y; x_{1}, \ldots, x_{n})$ under the obvious mapping $FJ[X] \to FJ[X]$), and take $h(x_{1}, \ldots, x_{n}, y) = U_{y}g(x_{1}, \ldots, x_{n}, y)$. Then, for any $z \in X$ we have
\[ h(z; x_{1}, \ldots, x_{n}, y) = P_{y}P_{z}g(z; x_{1}, \ldots, x_{n}, y) = P_{y}P_{z}f(P_{2}y; x_{1}, \ldots, x_{n}) \]
\[ = P_{y}f(y; P_{z}x_{1}, \ldots, P_{z}x_{n}) \]
by 0.19.

Now, if $a_{1}, \ldots, a_{n}, b \in V^{\sigma}$ and $c \in V^{-\sigma}$, computing in $T(V)$, we have
\[ h(c; a_{1}, \ldots, a_{n}, b) = P_{b}P_{c}f(P_{c}b; a_{1}, \ldots, a_{n}) = 0, \]
since $f$ is an identity of the homotope $V^{\sigma}(Q, b)$ (hence of $T(V)^{(P, b)}$).

Similarly, if $a_{1}, \ldots, a_{n}, b \in V^{-\sigma}$ and $c \in V^{\sigma}$,
\[ h(c; a_{1}, \ldots, a_{n}, b) = P_{b}f(b; P_{c}a_{1}, \ldots, P_{c}a_{n}) = 0, \]
since $f$ is an identity of the homotope $V^{\sigma}(b)$ (hence of $T(V)^{(b)}$). Therefore, $h$ is an identity of every homotope of $V$.

0.21. The fact that a Jordan system $J$ satisfies a homotope-PI means that all homotopes, hence all local algebras, satisfy a given identity. Often, we are interested in a weaker assertion, the existence of some $a \in J$ for which the local algebra $J_{a}$ is PI. We call such an element a PI-element, and write $\text{PI}(J)$ for the set of PI-elements of $J$ ($\text{PI}(V) = (\text{PI}(V^{+}), \text{PI}(V^{-}))$ if $J = V = (V^{+}, V^{-})$ is a Jordan pair). Thus, the fact that $J$ has a nonzero PI-element can be abbreviated to $\text{PI}(J) \neq 0$. We recall here the main results of Montaner (1999).
Let $J$ be a nondegenerate Jordan system. Then $\text{PI}(J)$ is an ideal of $J$.

A Jordan system $J$ is said to be rationally primitive if it is primitive and has a nonzero PI-element. This is the Jordan analogue of strongly primitive associative algebras. Rational primitivity is characterized in the following analogue of Amitsur's theorem on generalized identities (Montaner, 1999, Theorems 4.5 and 4.6)

**Theorem 0.23.** Let $J$ be a Jordan system. The following are equivalent

(a) $J$ is rationally primitive.
(b) $J$ is strongly prime and $\text{Soc}(T) = \text{PI}(T) \neq 0$.
(c) $J$ is strongly prime and the local algebra at some nonzero element is a simple unital PI-algebra.

As a consequence one has an analogue for Jordan systems of Kaplansky's theorem, with homotope polynomial identities on Jordan systems playing the role of polynomial identities on algebras.

**Theorem 0.24.** Let $J$ be a primitive Jordan pair or triple system.

(i) If the local algebra at each element of $J$ is PI, then $J$ is simple equal to its socle.
(ii) If $J$ satisfies a homotope-PI, then $J$ is simple with finite capacity.

1. PRIMITIZERS VS. WEAK PRIMITIZERS

It has been proved in Anquela and Cortés (1998) that primitivity and weak primitivity are equivalent notions, however we will need a more detailed information concerning primitizers. In this section we examine the relationship between primitizers and weak primitizers, and show how a weak primitizer gives rise to a primitizer which is closely related to it, and in particular has the same modulus. We first introduce some notation:

Let $J$ be a Jordan triple system, let $a, b$ be elements of $J$, and let $K$ be an inner ideal of $J$ which is weakly $(a, b)$-modular. We set:

$$K_0 = \{x \in K \mid \{x, b, a\} \in K\},$$
By the remark above and 0.12 it suffices to prove it for a triple

\[ K_1 = \{ x \in K \mid \{x, J, (a - P_h a)\} \subseteq K \}. \]

Notice that the definitions of \( K_0 \) and \( K_1 \) depend on the elements \( a \) and \( b \). These elements will be fixed throughout, so that there will be no ambiguity.

We give a similar definition for Jordan pairs. Let \( V \) be a Jordan Pair, and let \( K \subseteq V^\sigma \) be an inner ideal which is weakly \( (a^+, a^-) \)-modular for some \( a^\sigma \in V^\sigma \), which will be fixed throughout. We set \( K_0 = \{ x \in K \mid \{x, a^\sigma, a^\sigma\} \subseteq K \} \) and \( K_1 = \{ x \in K \mid \{x, V^\sigma, a^\sigma - Q_0 a^\sigma\} \subseteq K \} \).

Note that if, say, \( \sigma = + \), \( K \oplus V^- \) is a weakly \( (a^+, a^-) \)-modular inner ideal of \( T(V) = V^+ \oplus V^- \), and \( (K \oplus V^-)_i = K_i \oplus V^- \), for \( i = 0, 1 \).

**Lemma 1.1.** \( K_0 \) is a weakly \( (a, b) \)-modular inner ideal, and satisfies \( \{K_0, a, b\} \subseteq K_0 \), i.e. \( (K_0)_0 = K_0 \).

**Proof:** By the remark above and 0.12 it suffices to prove it for a triple system \( J \).

We first show that \( K_0 \) is inner. If \( x \in K_0 \), \( y \in J \), from JP12:

\[ \{P_a y, b, a\} = \{x, y, \{x, b, a\}\} - P_a \{y, a, b\} \in K, \]

since \( \{x, b, a\} \in K \).

Next, we prove that \( K_0 \) is weakly \((a, b)\)-modular. If \( y \in J \), \( \{B_{a, b}, y, b, a\} = L_{a, b} y - L_{a, b}^2, y + L_{a, b} y - L_{a, b}^2, b + P_{a, b} y = L_{a, b} y - L_{a, b}^2, b + P_{a, b} L_{a, b}, b \) (since \( L_{a, b} y = P_{a, b} L_{a, b}, b \) by JP1 twice) \( = B_{a, b} y - P_{a, b} L_{a, b}, b \). Therefore \( B_{a, b} J \subseteq K_0 \).

Now, take a unital hull \( J^{[b]} \) of the \( b \)-homotope. Computing in that algebra we have, for all \( x \in K \), \( y \in J \), \( \{x, y, 1 - a\} \circ (1 - a) = V_{1 - a} U_x, 1 - a = U_x, 1 - a V_{1 - a} + U_{1 - a} V_x - V_x U_{1 - a} \) (by Q18 after the substitution \( a = 1 - a, b = 1 \) and \( c = x \)). Now, since \( U_{1 - a} = B_{a, b} \) and \( U_x, 1 - a = L_{x, b - p}, b = P_{x, a} \) in \( J \), it follows from weak modularity of \( K \) that \( U_{1 - a} J + U_x, 1 - a J \subseteq K \), hence \( \{x, y, 1 - a\} \circ (1 - a) \in K \). Since \( \{x, y, 1 - a\} = U_x, 1 - a \in K \), this means that \( \{x, y, 1 - a\} \circ a \in K \), i.e. \( \{U_x, 1 - a\}, b, a \in K \), taking the brackets in the pair \( J \). Therefore, \( U_x, 1 - a J \subseteq K_0 \) which, together with \( B_{a, b} J \subseteq K_0 \), gives weak \((a, b)\)-modularity of \( K_0 \).

Finally, we have \( (K_0)_0 = K_0 \). Take \( x \in K_0 \), then \( \{x, b, a\}, b, a \) \( = 2 P_{a, b} x + \{x, b, a\}, a \) (by JP9). Now, \( P_{a, b} x = B_{a, b} x - x + \{x, b, a\} \in K \).
and \( \{x, P_{b\alpha}a, a\} = \{x, b, a\} - \{x, P_{\beta\alpha}a, a\} \subset K \) by weak modularity and the definition of \( K_0 \). Thus \( \{x, b, a\} \in K \), hence \( \{K_0, a, b\} \subseteq K_0 \), and \( (K_0)_b = K_0 \).

We now turn to the proof of the corresponding result for \( K_1 \).

**Lemma 1.2.** \( K_1 \) is a weakly \((a, b)\)-modular inner ideal, and satisfies \((K_1)_1 = K_1 \).

**Proof.** Again, it suffices to prove the result in case \( J \) is a triple system.

We first show that \( K_1 \) is inner. Take \( x \in K_1 \) and \( y, z \in J \). Then, \( \{P_x, y, z, a - P_{b\beta}\} = \{x, y, x, z, a - P_{b\beta}\} - P_x \{y, a - P_{b\beta}, z\} \) (by JP11) \( \in K \), since \( \{x, z, a - P_{b\beta}\} \in K \).

We next examine the weak \((a, b)\)-modularity of \( K_1 \). For \( y, z \in J \) we have \( \{P_{a\beta}y, z, a - P_{b\beta}\} = \{y, z, a - P_{b\beta}\} - \{y, b, a\} \cdot (a - P_{b\beta}) + \{P_{b\beta}y, z, a - P_{b\beta}\} \). Now we have the equalities:

\[
\{y, z, P_{a\beta}\} = \{y, z, a\} \cdot (a - P_{b\beta}) - P_x \{y, z, b\} \quad \text{(by JP11)},
\]
\[
\{y, b, a\} \cdot (a - P_{b\beta}) = \{y, b, P_{a\beta}z\} + \{y, b, a\} \cdot (a - P_{b\beta}) \quad \text{(by JP11)},
\]
\[
\{y, b, a\} \cdot (a - P_{b\beta}) = \{y, P_{a\beta}P_{b\beta}, a\} + \{y, b, a\} \cdot (a - P_{b\beta}) \quad \text{(by JP19)},
\]

and

\[
\{P_{a\beta}P_{b\beta}y, z, P_{a\beta}\} = \{P_{a\beta}P_{b\beta}y, P_{a\beta}z, b\} = \{P_{a\beta}P_{b\beta}y, b, P_{a\beta}z\} \quad \text{(by JP1 twice)}.
\]

Therefore,

\[
\{B_{a\beta}y, z, a - P_{b\beta}\} = \{y, z, a\} - \{y, z, a\} \cdot (a - P_{b\beta}) + \{y, b, a\} \cdot (a - P_{b\beta}) - \{P_{a\beta}P_{b\beta}y, z, a\}
\]

\[
= \{y, b, a\} \cdot (a - P_{b\beta}) + \{y, b, a\} \cdot (a - P_{b\beta}) - \{y, P_{a\beta}P_{b\beta}y, z, a\}
\]

\[
= \{y, b, P_{a\beta}z\} - \{y, b, P_{a\beta}z\} + \{y, b, P_{a\beta}z\} + \{y, b, P_{a\beta}z\}.
\]

We have the equalities:

\[
\{P_{a\beta}P_{b\beta}y, z, a\} = \{P_{a\beta}z, P_{b\beta}y, a\} \quad \text{(by JP4)},
\]
and

\[ \{y, P_b P_a z, a\} + \{P_a z, P_b y, a\} = \{\{y, b, P_a z\}, b, a\} \quad \text{(by linearized JP2)}. \]

Hence \( \{y, P_b P_a z, a\} + \{P_a z, P_b y, a\} = \{\{y, b, P_a z\}, b, a\} \), and

\( \{B_a, b\}, z, a - P_a b\} = B_a, b(\{y, z, a\} - \{y, b, P_a z\}) \in K. \)

Now, take \( x \in K_1 \) and \( y, z \in J \), we will prove

1. \( \{x, b, y\} - \{x, P_b y, a\}, z, a - P_a b\} \in K. \)

By JP15, we have:

2. \( \{x, b, y\} - \{x, P_b y, a\}, z, a - P_a b\} \)

\[ = \{x, b, y, z, a - P_a b\} + \{y, b, x, z, a - P_a b\}\]

\[ - \{x, z, \{y, b, a - P_a b\}\} - \{\{x, P_b y, a, z\}, a - P_a b\}\]

\[ + \{a, P_b y, \{x, z, a - P_a b\}\} - \{x, z, \{a, P_b y, a - P_a b\}\}. \]

Now,

3. \( \{x, b, y, z, a - P_a b\} - \{x, P_b y, a, z\}, a - P_a b\} \in \{K_1, J, a - P_a b\} \subseteq K. \)

4. \( \{y, b, x, z, a - P_a b\} - \{a, P_b y, \{x, z, a - P_a b\}\} \in K, \) by weak modularity of \( K. \)

5. \( \{x, \{y, b, a - P_a b\}\} - \{x, z, \{a, P_b y, a - P_a b\}\} = \{x, z, \{y, b, a - P_a b\}\} \notin K, \) since \( K \) is inner, \( x \in K, \) and

\( \{y, b, a - P_a b\} - \{a, P_b y, a - P_a b\} = \{y, b, a\} - \{y, b, P_a b\} - 2P_a P_b y + \{a, P_b y, P_a b\} = \{y, b, a\} - \{y, b, a\} + P_a P_b y \)

(\( b, y, \) and \( P_a P_b y \) by JP8 and \( P_a P_b y \) by JP1 twice) = \( B_a, b\{y, b, a\} \in K \) by weak modularity of \( K. \)

And (1) follows from (2), (3), (4) and (5).

Finally, take \( x \in K, \) and \( y, z \in J. \) We have \( \{x, y, a - P_a b, z, a - P_a b\} = B_a, bP_a\{y, x, a\} - \{x, y, B_a, bP_a z\} \in K, \) and therefore \( \{K, J, a - P_a \} \in K_1, \) hence \( (K_1) = K_1. \)

**Lemma 1.3.** With the previous notations, we have

(a) \( \{(K_0)\}, a, b\} \subseteq (K_0)\), hence \( ((K_0)\) = (K_0)\).
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(b) \(\{(K_0)_0, J, a - P_a b\} \subseteq (K_1)_0\) hence \((K_1)_0 = (K_1)_0\).

(c) If \(K\) complements nonzero ideals, then both \(K_0\) and \(K_1\) complement nonzero ideals.

Proof. (a) Take \(x \in (K_0)_1\), we must prove that \(\{x, b, a\} \in (K_0)_1\), i.e. that \(\{x, b, a, y, a - P_a b\} \subseteq K_0\) for all \(y \in J\).

Now, from JP11 and JP19 we obtain: \(\{x, b, a\} = \{x, b, P_a b, b, x, y\}\) and \(\{x, b, a, y, P_a b\} = P_a P_b \{a, y, x\} + \{a, b, y, P_a b\}\).

Hence

\[
\{x, b, a\}, y, a - P_a b = (\{x, b, P_a y\} - \{x, P_p P_b y, a\}) + P_a \{b, x, y\} - P_a P_b \{a, y, x\}.
\]

Now, again by JP11, \(P_a \{b, x, y\} = \{a, b, \{x, y, a\}\} - \{P_a b, y, x\}\), hence

\[
\{x, b, a\}, y, a - P_a b = (\{x, b, P_a y\} - \{x, P_p P_b y, a\}) - \{x, y, a\} - a, b, \{x, y, a\} + P_a P_b \{x, y, a\} + \{x, y, a - P_a b\}.
\]

Since \(\{x, b, P_a y\} - \{x, P_p P_b y, a\}\) and \(\{x, y, a\} - \{a, b, \{x, y, a\}\} + P_a P_b \{x, y, a\}\) belong to \(K_0\) by weak modularity, and \(x \in (K_0)_1\), we obtain \(\{x, b, a\}, y, a - P_a b\} \subseteq K_0\) because \(x \in (K_0)_1\).

(b) Take \(x \in (K_0)_0\), and \(y \in J\), and let us see that \(\{x, y, a - P_a b\}, b, a\} \subseteq (K_1)_0\) hence \(\{x, y, a - P_a b\} \equiv (K_1)_0\).

From JP11 we get \(\{a, b, \{x, y, a\}\} = P_a \{b, x, y\} + \{P_a b, y, x\}\) and \(P_a \{b, x, y\} = \{a, y, \{x, b, a\}\} - \{P_a y, b, x\}\), hence

(i) \(\{x, y, a\}, b, a\} = \{x, y, P_a b\} + \{a, y, \{x, b, a\}\} - \{P_a y, b, x\}\).

Now, from linearized JP2,

\[
\{x, b, P_a b, y, a\} = -\{x, b, P_a b, y, a\} + \{x, y, P_a b, b\}, a\} + P_a \{y, x, b\}, a\}.
\]

On the other hand, we have

\[
\{x, y, P_a b, b, a\} = \{x, y, a, P_a a, a\} (by JP2) = \{x, y, P_a a, P_a a\} + P_a \{P_a a, x, y\} (by JP11),
\]

\[
\{x, y, P_a b, b, a\} = \{x, y, a, P_a a, a\} (by JP2) = \{x, y, P_a b, a\} + \{x, y, P_a a, P_a a\} (by JP10).
\]
there are
implies
We set
and
Next, we prove that
Therefore,
(ii) \{x, b, P_b b, y, a\} = \{x, y, P_a b, a\} + \{x, b, a\} + \{y, P_a b, a\} - \{x, P_b b, a\},

Thus, (i) and (ii) yield
Now, \(P_b b - P_a b a - x = P_b b - B_{a, b, a}\), hence \(\{x, x, P_b b - P_a b a\} \in K_1\) since \(x \in (K_1)_0 \subseteq K_1\) and \(K_1\) is weakly modular, and \(\{x, b, a\}, y, a - P_b b\) - \(\{x, b, P_b b, a\} - \{x, P_b b, a\}\) \(\in K_1\) by weak modularity of \(K_1\). Thus \(\{x, x, a - P_b b, b, a\} \in K_1\), hence \(\{x, y, a - P_b b\} \in (K_1)_0\) for all \(y \in J\), and \(x \in ((K_1)_0)_1\).

(c) Let \(I\) be a nonzero ideal of \(J\), and suppose that \(J = K + I\). Then, there are \(k \in K\) and \(y \in I\) such that \(a = y + k\). We have \(\{P_b b, b, a\} = \{k, P_b k, a\} = 2P_b b - (\{k, b, k\} - \{k, P_b k, a\}) \in K\) by weak \((a, b)\)-modularity. Thus, \(P_b b \in K_0\). Now \(P_b b = P_b b + y'\) with \(y' = \{k, b, y\} + P_b b \in I\), and \((P_b b, b)\) is a weak modulus for \(K_0\) by 0.8, hence \((K_0 + I) = J\).

Next, we prove that \((K_1) + I = J\). By what we have just proved, we can assume that \(a = y + k\) with \(y \in I\) and \(k \in K_0\). Now, for any \(x \in J, P_b b, x, a - P_b b = \{P_b b, x, a\} - \{P_b b, x, P_b b\}\), and \(P_b b, x, P_b b = \{k, P_b k, x, a\} + P_k P_b P_x + P_k P_b P_x - P_k P_b P_x\) by JP3. On the other hand, \(P_k P_b b, x, a = \{k, b, k, x, a\} - P_k b, b, x, a\) by JP12, hence \(P_k P_b b, x, a - P_b b = \{k, b, k, x, a\} - \{k, P_b k, x, a\} - P_k b, b, a, x\) - \(P_k P_b P_x - P_k P_b P_x - P_k P_b P_x - P_k P_b P_x - P_k P_b P_x - \{P_k b, b, a, x\} \in (K_1 - b - P_k P_x) + P_k J + P_k J + P_{(k, a, b)} J + B_{a, b} J + (K_0, a, b) \subseteq K\). Therefore \(P_b b \in K_1\), and \(P_b b \in K_1 + I\) gives as before \(K_1 + I = J\).

From this lemma it follows that \((K_0)_1 (K_1)_0\). Indeed, \((K_0)_1 \subseteq K_1\) implies \((K_0)_1 \subseteq K_0\), hence \((K_0)_1 \subseteq (K_0)_0\), and similarly, \((K_1)_0 \subseteq (K_0)_1\). We set \(K = (K_0)_1 + \Phi(a - P_b b)\).

**Proposition 1.14.** With the previous notations, \(K\) is an inner ideal which is \((a, b)\)-modular which is proper if \(K\) is. Moreover, if \(K\) is a weak primitizer, then \(K\) is a primitizer, and if \(K\) is PI in some homotope \(J^{(c)}\), then \(K\) is PI in \(J^{(c)}\) too.
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Proof. We claim that \( P_{K}J \subseteq (K_0)_1 \). Indeed, \( P_{K}J \subseteq P_{(K_0)_1}J + \{(K_1)_0 \} \), \( J, a - P_{b}J \), \( a - P_{b}J \), and \( P_{(K_0)_1}J \), \( J, a - P_{b}J \). Also, \( a - P_{b}J = P_{b}x - \{a, x\} \), \( P_{(K_0)_1}J \), \( J, a - P_{b}J \). Therefore \( P_{a- p}J \subseteq (K_1)_0 \), and this proves the claim. As a consequence, \( K \) is an inner ideal. Now, if \( K = J \), then \( a(2, b) \in P_{J}P_{K}J \subseteq K \), and \( K = J \) by 0.7 and 0.8.

Since \( (K_1)_0 \) is weakly \( (a, b) \)-modular and has \( (K_1)_0 \), \( b, a \) \( (K_0)_1 \) by 1.3, to prove that \( K \) is \( (a, b) \)-modular it suffices to prove that \( \{a - P_{b}J, b, a\} \) and \( \{a - P_{b}J, b, x\} \) belong to \( \bar{K} \) for all \( x \in J \). Now, \( a - P_{b}J, b, a \) \( = 2P_{b}J - \{a, b\} \), \( b, a \) \( = 2P_{b}J - 2P_{a}J/a \), \( b, a \) \( \subseteq (K_0)_1 \) by \( \{a - P_{b}J, b, a\} \) belong to \( \bar{K} \) for all \( x \in J \). Thus, \( \bar{K} \) is \( (a, b) \)-modular.

If \( K \) is a weak primitizer, then it complements nonzero ideals, hence \( (K_0)_1 \) complements nonzero ideals by 1.3, and so does \( \bar{K} \). Thus \( \bar{K} \) is a primitizer.

Finally, suppose that \( K \) satisfies the PI \( f(x_1, \ldots, x_n) = 0 \) in the algebra \( \bar{K}^{(c)} \). We have proved that \( P_{K}J \subseteq (K_0)_1 \subseteq K \), hence \( (K_0)_1 \subseteq K \), and \( \bar{K} \) satisfies the PI \( f(x_1^2, \ldots, x_n^2) = 0 \)

As a consequence we have a new proof of Theorem 4.17 of Anquela and Cortés (1998) which does not make use of the structure theory.

Corollary 1.5. A Jordan pair or triple system is primitive if and only if it is weakly primitive.

2. MODULARITY IN PAIRS WITH FINITE CAPACITY

In this section we consider (weakly) modular inner ideals in Jordan pairs with finite capacity. We begin with the more general situation, which will also be needed in latter sections, where the inner ideal is (weakly) modular at a regular element.

Lemma 2.1. Let \( V \) be a Jordan pair, \( \sigma = \pm \), \( (a, b) \in V^\sigma \times V^{-\sigma} \), and \( K \subseteq V^\sigma \) be an inner ideal. Suppose that \( b \) is regular, and take \( e = (e^+, e^-) \) an idempotent with \( e^{-}\sigma = b \). Then:

(a) If \( K \) is weakly \( (a, b) \)-modular, then:
Lemma 2.2. Let V be a Jordan pair, $\sigma = \pm (a,b) \in V^\sigma \times V^{-\sigma}$, and $K \subseteq V^\sigma$ be an $(a,b)$-modular inner ideal. Suppose that $b$ is regular, and take $e = (e^+, e^-)$ an idempotent with $e^- = b$. If $M \subseteq V^\sigma$ is an $(e^+, e^-)$-modular inner ideal, and $K \subseteq M$, then $M$ is $(a,b)$-modular. Therefore, $K$ is maximal $(a,b)$-modular if and only if it is maximal $(e^+, e^-)$-modular.

Proof. As before we can assume $\sigma = +$. 

(i) $V^\sigma_0(e) \subseteq K$ and 
(ii) $\{V^\sigma_1(e), b, K\} \subseteq K$.

Moreover, any $K$ satisfying i and ii is weakly $(e^\sigma, e^{-\sigma})$-modular.

(b) If $K$ is $(a,b)$-modular, then it satisfies i, ii, and

(iii) The Peirce 1-component with respect to $e$ of any element of $K$ belongs to $K$.

Moreover, any $K$ satisfying i, ii and iii is $(e^\sigma, e^{-\sigma})$-modular.

Proof. Taking the opposite pair if necessary, we can assume $\sigma = +$.

(a) If $x \in V^+_0$, $B_{a,b}x = x - \{a, b, x\} + Q_aQ_bx \in K$ by weak modularity. Now $\{a, b, x\} \in \{a, e^-, V^+_0(e)\} = 0$, and $Q_aQ_bx \in Q_aQ_e^{-}V^+_0 = 0$, hence $x \in K$.

Next, if $x \in V^+_1(e)$, for any $k \in K$, we have $\{k, b, x\} - \{a, Q_bx, k\} \subseteq K$ by weak modularity, but $Q_bx \in Q_aV^+_1(e) = 0$, hence $\{x, b, k\} \subseteq K$.

Now, suppose that $K$ satisfies i and ii. If $x \in V^+$, take $x = x_0 + x_1$ its Peirce decomposition with respect to $e$. If $k \in K$, $(e^+, Q_e^{-}x) = \{e^+, e^-, x\} = \{x, Q_e^{-}e^+, k\}$ (by JP7) = $\{x, e^-, k\} - \{x, e^+, k\} = \{x_2, e^-, k\} - \{x_0, e^-, k\}$, hence $\{x, e^-, k\} - \{e^+, Q_e^{-}x, k\} = \{x_1, e^-, k\} + 2\{x_0, e^-, k\} = \{x_1, e^-, k\}$ (since $\{x_0, e^-, k\} \in \{V^+_0, e^-, V\} = 0$) $\subseteq K$ by ii. Since $B_{e^+, e^-}x = V^+_0 \subseteq K$ by i, this gives weak $(e^+, e^-)$-modularity of $K$.

(b) Properties i and ii hold by (a). To prove iii, take $k \in K$, and write $k = k_0 + k_1 + k_2$ for its Peirce decomposition with respect to $e$. By modularity we have $\{a, b, k\} = \{k, e^+, a\} \subseteq K$, and $\{k, b, e^\pm\} - \{k, Q_ae^\pm, a\} = \{k, e^+, e^-\} - \{k, Q_ae^+, e^-\} = \{k, e^+, e^\pm\} - \{k, e^+, a\} \subseteq K$, hence $2k_2 + k_2 = \{k, e^+, e^-\} \subseteq K$. Also, since $k_0 \in K$ by i, $k_2 + k_2 = k \in K$, hence $k \subseteq K$.

Now, if $K$ satisfies properties i, ii, and iii, $K$ is weakly $(e^+, e^-)$-modular by (a), $(K, e^+, e^-) \subseteq K$ by i and iii, and $e^+ - Q_ae^- = 0 \in K$, hence $K$ is $(e^+, e^-)$-modular.
From \((a,b)\)-modularity of \(K\) we get \(B_{a,b}V \subseteq K \subseteq M\) and \(a - Q_a b \in K \subseteq M\). Now, \(B_{a,b} e^+ = B_{a,c} e^+ = e^+ - \{a, e^- a, e^+\} - Q_a Q_c e^+ = e^+ - \{a, e^-, e^+\} + Q_a e^- = 2 e^+ - \{a, e^-, e^+\} + (a - e^+) + (Q_a e^- - a) = \{e^+, ae^- e^+, a - e^-\} + (Q_a b - a) \in K\). Thus, \(e^+ - ae^-, e^+\} - (e^+ - a) \in K\). From this we get \(e^+ - Q_a e^- a = Q_a e^- (e^+ - a) = B_{e^+, e^-}(e^+ - a) + \{e^+ - a, e^- e^+\} - (e^+ - a) \in K\). Thus, if we write \(a = a_0 + a_1 + a_2\) for the Peirce decomposition of \(a\) with respect to \(e, w = a_2 - e^+ \in K \subseteq M\).

Now, for \(m \in M\), \(\{a, b, m\} = \{a, e^- m, \} + \{a, e^-, m\}\) (since \(\{a_0, a_1, m\} \in \{V e^+_0, V e^- (e), V^+\} = \{e^+, e^- m, w, e^- m\} + \{a_1, e^- m, \} \in \{e^+, e^- M\} + Q_m V^+ + \{V e^+_0, e^- M\} \subseteq M\) by 2.1. Next, if \(x \in V^+\) and \(m \in M\) we have \(\{a_1, Q_0 x, m\} = \{a_1, e^-, x, e^-, m\}\) (by identity JP7). Now, \(\{a_1, e^-, x, e^- m\} \in \{e^+, e^- V^+\}, e^- M\) \(\subseteq \{V e^+_0, e^- M\} = \{e^+, e^- M\} \subseteq M\) by 2.1(ii). Thus \(\{a_1, Q_0 x, m\} \in M\).

Then, \(\{x, b, m\} = \{a, Q_0 b, m\} = \{x, e^- m\} - \{a_2, Q_0 x, m\} - \{e^+, Q_0 x, m\} - \{w, Q_0 x, m\} - \{a_1, Q_0 x, m\} \in M\) since we have \(\{a_1, Q_0 x, m\} \in M\), \(\{w, Q_0 x, m\} \in Q_m V^+ \subseteq M\), and \(\{x, e^- m\} = \{e^+, Q_0 x, m\} \in M\) by \((e^+, e^-)\)-modularity of \(M\). Therefore \(M\) is \((a,b)\)-modular.

The last assertion is straightforward from this.

It follows from the previous lemmas that the study of (maximal) modularity of inner ideals in Jordan pairs with finite capacity reduces to the study of (maximal) modularity at idempotents (although we still have to consider arbitrary weak moduli). We next show that we can change the idempotent. Recall that two idempotents \(e, f\) of a Jordan pair \(V\) are associated if their Peirce spaces coincide \(V e = V f, i = 0, 1, 2\), and they are orthogonal if \(e, f = V_0 (e) = V_2 (f), i = 0, 1, 2\), and in this case, \(f = V_0 (e)\) and \(e+f = \) is again an idempotent.

**Lemma 2.3.** Let \(V\) be a Jordan pair, and let \(e = (e^+, e^-)\) and \(f = (f^+, f^-)\) be idempotents of \(V\). Then:

(a) if \(e\) and \(f\) are associated idempotents, every (weakly) \((e^+, e^-)\)-modular inner ideal is (weakly) \((f^+, f^-)\)-modular.

(b) if \(e\) and \(f\) are orthogonal idempotents, every (weakly) \((e^+, e^-)\)-modular inner ideal is (weakly) \((e^+, f^+)\)-modular.

**Proof.** (a) We use the characterization 2.1. Take a weakly \((e^+, e^-)\)-modular inner ideal \(K\). Then 2.1(i) follows from \(B_{e^+, e^-} V^+ = V_0^+(e) = V_0^+(f) = B_{f^+, f^-} V^+\), and, if \(K\) is \((e^+, e^-)\)-modular, 2.1(iii) for \((f^+, f^-)\) follows from 2.1(iii) for \((e^+, e^-)\) and the fact that the Peirce components with
respect to $e$ coincide with the Peirce components with respect to $f$. Now, to prove 2.1(ii), take $x \in V_1^+(f) = V_1^+(e)$, and $k \in K$. Then $f^+ = Q_e Q_e f^-$, hence \( \{x, f^-, k\} = \{x, Q_e Q_e f^-, k\} = \{\{x, e^+, Q_e f^-, e^-, k\} + \{Q_e f^-, Q_e - x, k\} \) (by JP7) $= \{\{x, e^+, Q_e f^-, e^-, k\} \) (because $Q_e - x \in Q_e - V_1^+ e = 0 \} \in \{\{V_1^+(e), V_2^-(e) V_2^+(e)\}, e^-, K\} \subseteq \{V_1^+(e), e^-, K\} \subseteq K$ (by 2.1(ii)).

(b) Take a (weakly) $(e^+, e^-)$-modular inner ideal $K$. By Loos (1975, Theorem 5.14), $V_0(e+f) = V_0(e) \cap V_0(f)$, hence $B_{e+f,e^-+f^-} = V_0(e+f) V_0(e+f) \subseteq V_0(e) \subseteq K$. This proves 2.1(i). Next, again by Loos (1975, Theorem 5.14), $V_1(e+f) = V_1(e) \cap V_1(f) + V_0(e) \cap V_1(f)$. Take $x \in V_1^+(e+f)$ and write $x = x_1 + x_0$ with $x_1 \in V_1^+(e) \cap V_0(f)$, and $x_0 \in V_0^+(e) \cap V_1^+(f)$. Now, if $k \in K$, $\{x_0, e^- + f^-, k\} \in \{V_0(e), V^- K \subseteq Q_K V^- K$, and $\{x_1, e^- + f^-, k\} \in \{V_1^+(e), e^-, K\} \subseteq K$ (by 2.1(ii)). Therefore $\{V_1^+(e+f), e^- + f^-, K\} \subseteq K$.

Finally, if $K$ is $(e^+, e^-)$-modular, and $k \in K$, we have $\{e^+ + f^+, e^- + f^-, k\} = \{e^+, e^- k\} + \{f^+, f^-, k\} \subseteq K$ (by 2.1(iii) and $f^+ \subseteq K$), and this, together with $V_0(e+f) \subseteq K$, implies that the Peirce 1-components with respect to $e+f$ of the elements of $K$ belong to $K$.

We next show that these changes of moduli preserve maximality.

**Lemma 2.4.** Let $V$ be a Jordan pair, and let $e = (e^+, e^-)$ and $f = (f^+, f^-)$ be idempotents of $V$. Then:

(a) If $e$ and $f$ are associated idempotents, every maximal (weakly) $(e^+, e^-)$-modular inner ideal is maximal (weakly) $(f^+, f^-)$-modular.

(b) If $e$ and $f$ are orthogonal idempotents, every maximal (weakly) $(e^+, e^-)$-modular inner ideal is maximal (weakly) $(e^+ + f^+, e^- + f^-)$-modular.

**Proof.** (a) is obvious since being associated is a symmetric condition for idempotents.

(b) Let $K$ be a (weakly) $(e^+, e^-)$-modular inner ideal of $V$ and suppose that $K \subseteq M$ where $M$ is a (weakly) $(e^+ + f^+, e^- + f^-)$-modular inner ideal of $V$. We claim that $M$ is (weakly) $(e^+, e^-)$-modular. First, we have $B_{e^+, e^-} V^+ K \subseteq M$. Now, $V_1(e) = V_1(e) \cap V_1(f) + V_0(e) \cap V_1(f)$ by Loos (1975, Theorem 5.14). Thus, if $x \in V_1^+(e)$ we can write $x = x_1 + x_0$ where $x_1 \in V_1^+(e) \cap V_1^+(f)$. Then, for any $m \in M$, $\{x_0, e^+, m\} = \{x_0, e^+ + f^-, m\} \in \{V_1^+(e+f), e^- + f^-, M\} \subseteq M$ by 2.1(ii). On the otherhand,
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\[ x_1 = \{x_1, f^-, f^+\} \] since \( x_1 \in V^\pm_1(f) \). Thus, \( \{x_1, e^-, m\} = \{x_1, f^-, f^+, e^-, m\} = \{f^+, \{f^-, x_1, e^+, m\} + \{x_1, f^-, e^+, m\} - \{f^+, e^-, \{x_1, f^-, e^+, m\} \} \) (by JP14) = \( \{f^+, \{f^-, x_1, e^+\}, m\} \) (since \( \{f^+, e^-, \, V^+\} \subseteq V_0^+(e), V_2^+(e), V^+ = 0 \) in \( M \) (since \( f^+ \in V_0^+(e) \subseteq K \subseteq M \)). Therefore, \( \{x, e^-, m\} = \{x_0, e^-, m\} + \{x_1, e^-, m\} \in M \), and this proves 2.1(ii). Now, if \( (e^+, e^-) \) is a modulus for \( K \), and not merely a weak modulus, \( M \) is \( (e^+ + f^+, e^- + f^-) \)-modular and \( m \in M \). \( \{e^+, e^-\} \) and \( \{e^+, f^+, e^- + f^-\}, m \) (since \( \{e^+, f^-, m\} = 0 \) \( \{e^+ + f^+, e^- + f^-\}, m \) - \( \{f^+, e^- + f^-\}, m \) in \( M \) (by 2.1(iii) and \( f^+ \in M \)). Since \( V_0^+(e) \subseteq M \), it follows that the Peirce 1-components of the elements of \( M \) are again in \( M \).

Assertion (b) readily follows from the claim.

Remark 2.5. Let \( V \) be a Jordan pair with finite capacity, \( e = (e^+, e^-) \) be an idempotent of \( V \), and \( K \subseteq V^\sigma \) be a (weakly) \( (e^\sigma, e^-^\sigma) \)-modular inner ideal. Since \( V \) has finite capacity, by Loos (1991b, Proposition 3) there is a frame \( \{e_1, \ldots, e_n\} \) and \( r \ge n \) such that \( e \) and \( e_1 + \cdots + e_r \) are associated idempotents. Set \( u = e_1 + \cdots + e_r, v = e_{r+1} + \cdots + e_n, \) and \( f = u + v \). Then \( K \) is (weakly) \( (u^\sigma, u^-^\sigma) \)-modular by 2.3(a) since \( e \equiv u \), and \( K \) is (weakly) \( (f^\sigma, f^-^\sigma) \)-modular by 2.3(b) since \( u \perp v \). Therefore \( K \) is (weakly) \( (f^\sigma, f^-^\sigma) \)-modular for a complete idempotent \( f = (f^+, f^-) \) (recall that an idempotent \( f \) is called complete if \( V_0(f) \) is contained in the Jacobson radical of \( V \)), which, for a Jordan pair with finite capacity means \( V_0(f) = 0 \). Moreover, if \( K \) is maximal (weakly) \( (e^\sigma, e^-^\sigma) \)-modular, it is maximal (weakly) \( (f^\sigma, f^-^\sigma) \)-modular by 2.4.

Modularity at a complete idempotent \( f \) as in the remark makes condition 2.1(i) superfluous since \( V_0(f) = 0 \). With more effort, we next show that condition 2.1(iii) is also superfluous in this case.

Proposition 2.6. Let \( V \) be a nondegenerate Jordan pair of finite capacity, and let \( e = (e^+, e^-) \) be a complete idempotent of \( V \). Then an inner ideal \( K \subseteq V^\sigma \) is \( (e^\sigma, e^-^\sigma) \)-modular if and only if it is weakly \( (e^\sigma, e^-^\sigma) \)-modular.

Proof. Clearly, if \( K \) is \( (e^\sigma, e^-^\sigma) \)-modular, it is weakly \( (e^\sigma, e^-^\sigma) \)-modular.

For the reciprocal it suffices to show that the Peirce 1-component with respect to \( e \) of any element from \( K \) is again in \( K \) since \( V_0(e) = 0 \). We assume \( e = + \), passing to the opposite pair if necessary.

Take \( k \in K \). Let us see that its Peirce 1-component is in \( K \). We do the proof in two steps:

Step 1. Suppose that the rank of \( k \) is 1.

If \( k = k_1 + k_2 \) is the Peirce decomposition of \( k \) relative to \( e \), we can assume that \( k_1 \neq 0 \neq k_2 \), since otherwise the result is trivial. By Loos
(1989, Lemma 5), \(k_1 = d^+\) can be extended to a division idempotent \(d = (d^+, d^-) \in V_2(e)\) such that \(V_2(d) \subseteq V_1(e)\). Moreover, \(Q_d d^- \in K.Q_d d^- = d^- + \{d^+, d^-, k_2\}\) (since \(Q_d k_2 \in Q_{V_1(e)} V_2(e) = 0\), and \(\{d^+, d^-, k_2\} \in V_1(d)\) (since \(Q_d k_2 \in Q_{V_1(e)} V_2(e) \subseteq V_0(e) = 0\) implies \(k_2 \in V_1(d) + V_0(d)\)). Also \(\text{rk}(Q_d d^-) \leq \text{rk}(k) = 1\) by Loos (1991a, Proposition 3(5)), so we can assume that \(k_2 \in V_1(d)\) substituting \(k\) by \(Q_d d^-\) if necessary.

By regularity, there exists an idempotent \(c = (c^+, c^-) \in V_2(e)\) with \(c^+ = k_2\). Let \(c^- = c_2 + c_1 + c_0\) be the Peirce decomposition of \(c^-\) relative to \(d\). Then, \(c_2 = Q_d Q_d c^- \in Q_d Q_{V_1(e)} V_2(e) \subseteq Q_d V_0(e) = 0\), hence \(c^- = c_1 + c_0\), and we have \(c^+ = Q_e c_1 + Q_e c_0 \in V_1(d)\), and \(Q_e c_0 \in Q_{V_1(e)} V_1(d) \subseteq V_2(d)\), hence, matching Peirce components, \(Q_e c_0 \neq 0\), and \(c^+ = Q_e c_1\). On the other hand, \(c^- = Q_e c^+ = Q_e c_1 + Q_e c_0 + c^+ + \{c_1, c^-\}, \) and \(Q_e c_1 \in Q_{V_1(e)} V_1(d) \subseteq V_1(d)\); \(Q_e c_0 + c^+ + \{c_1, c^-\} \in \{V_1(e), V_2(e), V_2(e)\} \subseteq V_2(e)\), so we can assume that \(c \in V_1(d)\).

Now, if \(\{c^+, c^-, d^+\} = 0\), we get \(d^- \in V_0(e)\), since we have \(Q_e c_0, Q_e d^- \in Q_{V_1(e)} V_1(e) = 0\). But then, \(c^- = \{d^-, d^-, d^+\} \notin \{d^-, V_0(e), V_2(e)\} = 0\), which is a contradiction since \(0 \neq k_2 = c^+ = Q_e c^-\).

Thus, \(\{c^+, c^-, d^+\} \neq 0\). Next, \(\{c^+, c^-, d^+\} = \{c^+, Q_e c^-, e^-, d^+\}\) (since \(c \in V_2(e)\) and \(V_2(e) = 0\)), \(\{c^+, e^-, \{Q_e c^-\}, e^-, d^+\}\) are \(\{c^+, e^-, Q_e c^-, e^-, d^+\}\) (by JP7) = \(\{c^+, e^-, Q_e c^-\}, e^-, d^+\}\) (since we have \(Q_e d^+ \in Q_e V_1(e) = 0\)), \(d^+\), \(V_1(e)\) (since \(Q_e c^-, e^-, d^+\) \(\in \{V_2(e), V_1(e)\} \subseteq V_1(e)\)). Now, \(\{c^+, e^-, V_1(e)\} \subseteq V_1(e), V_2(e), V_2(e) \subseteq V_1(e)\) \(\subseteq V_0(e) = 0\), hence \(\{c^+, e^-, d^+\} \notin \{c^+, e^-, V_1(e)\} = \{c^+, d^+, e^-, V_1(e)\} = \{k, e^-, V_1(e)\} \subseteq K\) (by 2.1(ii)).

Finally, \(0 \neq \{c^+, e^-, d^+\} \in \{V_1(d), V_1(d), V_2(d) \subseteq V_2(d) = Q_d V\},\) and since \(d^+ = k_1\) is simple, \(k_1 \in Q_{c^+, e^-, d^+} V \subseteq Q_K V \subseteq K\).

**Step 2.** Now take \(k \in K\) arbitrary (\(\text{rk}(k) \geq 1\)). By regularity there exists \(b \in V^-\) with \(Q_k b = k\). Write \(b = \sum u_i\) as a sum of rank one elements. Setting \(k_i = Q_k u_i \in K\), we have \(k = Q_k b = k_1 + \cdots + k_n\), and for each \(i\), \(\text{rk}(k_i) = \text{rk}(Q_k u_i) \leq \text{rk}(u_i) = 1\) by Loos (1991a, Corollary 1). Then, the Peirce 1-component of each \(k_i\) belongs to \(K \) by step 1, and thus, the Peirce 1-component of \(k\), which is the sum of those components, also belongs to \(K\).

**Remark 2.7.** Let \(V\) be a nondegenerate Jordan Pair with finite capacity, and let \(e = (e^+, e^-)\) be a complete idempotent. From 2.6 it follows that any inner ideal \(K\) containing \(V_1(e)\) is \((e^+, e^-)\)-modular, since \(\{V_1(e), e^-, V_0(e) \subseteq V_1(e) \subseteq K\).

We can drop the hypothesis of completeness of the idempotent using 2.3.
Corollary 2.8. Let \( V \) be a nondegenerate Jordan pair of finite capacity, and let \( e = (e^+, e^-) \) be an idempotent of \( V \). Then an inner ideal \( K \subseteq V^\sigma \) is \((e^\sigma, e^{-\sigma})\)-modular if and only if it is weakly \((e^\sigma, e^{-\sigma})\)-modular.

Proof. Obviously, modularity implies weak modularity.

Assume then that \( K \) is weakly \((e^\sigma, e^{-\sigma})\)-modular and, taking the opposite pair if necessary, that \( \sigma = + \). As in Remark 2.5 we can find orthogonal idempotents \( u = (u^+, u^-) \) and \( v = (v^+, v^-) \) such that \( e \approx u \), and \( f = u + v \) is complete, and \( K \) is weakly \((u^+, u^-)\)-modular and weakly \((f^+, f^-)\)-modular by 2.3. Then \( K \) is \((f^+, f^-)\)-modular by 2.6. On the other hand, if \( k \in K \), \( \{k, u^+, u^-\} = \{k, f^+, f^-\} - \{k, v^+, v^-\} \) (by orthogonality \( u \perp v \)) \( \in \{K, f^+, f^-\} + Q_K V^- \) (since \( V^+ \subseteq K \)).

Finally, we can drop the hypothesis of the modulus being an idempotent, thus proving Anquela and Cortés' conjecture for pairs with finite capacity.

Theorem 2.9. Let \( V \) be a nondegenerate Jordan pair of finite capacity, and let \( (a, b) \in V^\sigma \times V^{-\sigma} \). Then an inner ideal \( K \subseteq V^\sigma \) is \((a, b)\)-modular if and only if it is weakly \((a, b)\)-modular.

Proof. We can assume \( \sigma = + \) by taking the opposite pair if necessary. Complete \( b \) to an idempotent \( e = (e^+, e^-) \) with \( e^- = b \). Then \( K \) is weakly \((e^+, e^-)\)-modular by 2.1(a), hence \((e^+, e^-)\)-modular by 2.8.

Now, write \( a = a_0 + a_1 + a_2 \) for the Peirce decomposition of \( a \) relative to \( e \), and take \( x \in V^+_2(e) \). Setting \( z = Q_{a_1+a_0}x + \{a_2, x, a_1 + a_0\} = Q_u x + \{a_2, x, a_1\} \), we have \( Q_u x = Q_u x + z \) and \( z \in V^+_1(e) \).

Next, set \( y = Q_{e^-}xe \). We have \( \{e^+, x, a\} = \{e^+, Q_e y, a\} = \{e^+, e^-, y\} - \{y, Q_e e^+, a\} \) (by JP7) \( = 2\{y, e^-, a\} = \{y, e^-, a\} \) (since \( y \in V^+_2(e) \) = \( \{y, e^-, a\} \). Then \( e^+, x, a_2 = \{y, b, a\} - t \) where \( t = \{e^+, x, a_1 + a_0\} \in V^+_1(e) \).

Thus, \( Q_{e^-a_2}^2x = Q_{e^-a_2}^2x + Q_{e^-a_2}x = \{e^+, x, a_2\} = y + (Q_{e^-a_2}x - z) - \{y, b, a\} - t = y + Q_{e^-a_2}y - \{y, b, a\} + (t - z) \) (since \( x = Q_{e^-a_2}x = Q_{e^-}y = Q_{e^-}y + (t - z) \in K + V^+_1(e) \). Therefore \( Q_{e^-a_2}V^+ = Q_{e^-a_2}V^+_2(e) \subseteq K + V^+_2(e) \), hence \( e^+ - a_2 \subseteq Q_{e^-a_2}V^- \) (by regularity) \( \subseteq K + V^+_2(e) \). On the other hand, \( Q_{e^-}K \subseteq K \) by \((e^+, e^-)\)-modularity, hence \( e^+ - a_2 = Q_{e^-}Q_{e^-a_2}Q_{e^-}Q_{e^-}K \subseteq K \).
Now, if \( k \in K \), \( \{a, b, k\} = \{a_2, b, k\} + \{a_1, b, k\} + \{a_0, b, k\} = \{e^+, e^-, \}
\)
\( \{e^+ - a_2, b, k\} + \{a_1, e^-, k\} + \{a_0, b, k\} \in \{e^+, e^-\}, K\right) + Q_K V^- \langle e^-, K\rangle + Q_K V^- \subseteq K \) (by 2.1). Thus \( \{a, b, K\} \subseteq K \).

On the other hand, \( a_2 + a_1 - Q, b = -e^+ + 2a_2 + a_1 - Q, b + \left( e^+ - a_2 \right) = -\left( e^+ - \{e^+, e^-\}, a + Q, b \right) + \left( e^+ - a_2 \right) = -e^+ - e^+ + (e^+ - a_2) \subseteq K\), hence \( a - Q, b = (a_2 + a_1 - Q, b) + a_0 \in K \), and \( K \) is \( (a, b) \)-modular.

We next prove maximality of maximal-modular ideals in Jordan pairs with finite capacity.

**Theorem 2.10.** Let \( V \) be a nondegenerate Jordan pair with finite capacity, then any maximal-modular inner ideal is maximal among all inner ideals.

**Proof.** Let \( V \) be a nondegenerate Jordan pair with finite capacity, and let \( K \subseteq V^+ \) be an inner ideal (if \( K \subseteq V^- \) it suffices to consider the opposite pair) which is maximal \( (a, b) \)-modular for some modulus \( (a, b) \in V^+ \times V^- \). Take \( I = \text{Core}(K) \) the core of \( K \) (see 0.11). Clearly, \( V/I \) has finite capacity and \( K/I^+ \) is maximal \( (a + I^+, b + I^-) \)-modular in \( V/I \). Moreover, \( K \) will be maximal among all inner ideals of \( V \) if \( K/I^+ \) is maximal among all inner ideals of \( V/I \). Since obviously \( \text{Core}(K/I^+) = 0 \), we can assume that \( \text{Core}(K) = 0 \). Thus \( V \) is primitive, hence strongly prime (0.10), and since it has finite capacity, it is simple (Loos, 1975, Theorem 10.14).

By 2.2 and 2.5 we can assume that \( K \) is maximal \( (e^+, e^-) \)-modular for a complete idempotent \( e = \{e^+, e^+\} \). Now, if \( K \subseteq V_1^+ (e) \), it follows from 2.7 and the maximality of \( K \) that \( K = V_1^+ (e) \) is maximal among all inner ideals of \( V \). Thus we can assume that \( K \not\subseteq V_1^+ (e) \), and since all Peirce components relative to \( e \) of the elements of \( K \) lie again in \( K \), we can take \( 0 \neq k \in V_1^+ (e) \cap K \). Now, complete \( k \) to an idempotent \( c = \{e^+, e^+\} \subseteq V_2 (e) \), with \( e^+ = k \). By Loos (1991b, Proposition 3) there exists a frame \( \{e_1, \ldots, e_n\} \) in \( V_2 (e) \) and \( r \leq n \) such that \( e \approx e_1 + \cdots + e_r \). Then, \( e_r^+ \in Q, V^+ = Q, V^- \subseteq K \) and we can choose \( c = e_1 \), for a frame \( \{e_1, \ldots, e_n\} \) with \( e = e_1 + \cdots + e_r \). We consider the Peirce decomposition \( V = \bigoplus_{0 \leq f \leq g} V_f \) of \( V \) relative to the orthogonal system of idempotents \( (e_1, \ldots, e_n) \) (Loos, 1975, 5.14).

Take now a proper inner ideal \( M \subseteq V^+ \) with \( K \subseteq M \). We will show that \( M \) is \( (e^+, e^-) \)-modular.

Take \( x \in V_1^+ (e) \), and write \( x = x_0 + \cdots + x_n \) with \( x_0 \in V_1^+ (e) \cap (\bigcap_{0 \neq i} V_0^+ (e_i)) \), and let \( m \in M \). Note that \( V_0^+ = \{V_{01}^+, e^+, e^-, V_0^+, e^+, e^-, K\} \subseteq K \) by 2.1(ii), hence \( x_0 \in K \), and \( \{x_0, e^+, m\} \in Q_M V^- \subseteq M \).
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Next, since \( V \) is simple, \( e_1 \) and \( e_2 \) are connected (Peterson, 1978, Proposition 2). Thus there exists an element \( u \in V_1^+(e_1) \cap V_1^+(e_2) \) which is invertible in \( V_2(e_1 + e_2) \). Then \( Q_u \) induces a bijection between \( V_2^+(e_1) \) and \( V_2^+(e_2) \), so there is a unique \( z \in V_2^+(e_1) \) such that \( Q_u z = e_2 \). Now \( \{x_{02}, e^-, m\} = \{x_{03}, e^+, m\} = \{x_{02}, Q_u z, m\} = \{x_{02}, u, z\} \), \( u, m = \{z, Q_u x_{02}, m\} \) (by JP7) = \( \{x_{02}, u, z\} \), \( u, m \) (since \( Q_u x_{02} \in QV_1^+(e_1) \) \( = 0 \) \( \in \{V_2^{+}, V_{12}, V_{11}^+\} \), \( u, m \) \( \subseteq \{V_0^{+}, V^{-}\} \subseteq M \) \( \subseteq QMV_1^\circ \subseteq M \). Therefore \( \{x_{02}, e^-, m\} \in M \), and similarly, \( \{x_{0i}, e^-, m\} \in M \) for \( i = 2, \ldots, n \), hence \( \{x, e^-, m\} \in M \). Thus \( \{V_1^+(e^+), M\} \subseteq M, \) and \( M \) is weakly \( (e^+, e^-) \)-modular by 2.1, hence \( (e^+, e^-) \)-modular by 2.6.

Since \( K \) is maximal-modular for the modulus \( (e^+, e^-) \), it follows that \( K = M, \) and \( K \) is maximal among all inner ideals. ■

3. PRIMITIZERS AND LOCAL PI-ALGEBRAS

The study (Montaner, 1993, Anquela et al., 1993a, b) of maximal modular ideals of Jordan algebras required the distinction between PI and non-PI primitizers. Here we will need to make a distinction between primitizers (or weak primitizers) which are PI in any homotope and primitizers which are not.

We first recall the main result on algebras having a PI primitizer (Anquela et al., 1993a, Theorem 2.2) (see also Anquela and García, 2000), and give a proof of it that uses the ideas of Montaner (1999).

**Theorem 3.1.** If a Jordan algebra \( J \) has a PI-primitizer, then \( J \) is simple and unital.

**Proof.** Let \( \hat{J} \) be a tight unital hull of \( J \), and set \( \hat{K} = K + \Phi(1 - e) \), for a modulus \( e \) of \( K \), then \( \hat{K} \) is again a PI primitizer for \( J \) (Anquela et al., 1993a, Sec. 0). If \( J \) is simple, then \( J = \hat{J} \), since \( J \) is an ideal of \( \hat{J} \), and \( J \) is simple and unital. Therefore we can assume that \( J \) is unital.

If \( J \) is PI, the result follows from 0.17, so we can assume that \( J \) is not PI. Now, let \( f \) be an essential Jordan polynomial with \( f(K) = 0 \). For any nonzero ideal \( I \) of \( J \), \( J = K + I \), hence \( f(J) \subseteq f(K) + I = I \). Then \( f(J) \) is contained in every nonzero ideal of \( J \). But \( f(J) \neq 0 \), because \( J \) is not PI, hence the intersection of all nonzero ideals of \( J \) is nonzero, i.e., the heart of \( J \) is nonzero: \( \text{Heart}(J) \neq 0 \), and since each nonzero ideal of \( J \) contains a modulus for \( K \), we can assume \( e \in \text{Heart}(J) \).

If \( (1 - e)^2 = 0 \), then \( 1 = 2e - e^2 \in \text{Heart}(J) \), and \( \text{Heart}(J) = J \) is simple. Suppose then \( (1 - e)^2 \neq 0 \). Now, the mapping \( J^{(1 - e)^2} \rightarrow J \) given by
\(x \mapsto U_{1-e}x\) is a homomorphism whose kernel contains \(\text{Ker}(1-e)^2\), hence \(J_{(1-e)^2}\) is isomorphic to a quotient of the image \(U_{1-e} J \subseteq K\), which is PI. Therefore \(0 \neq (1-e)^2 \in \text{PI}(J)\). But \(\text{Heart}(J) \subseteq \text{PI}(J)\) since the latter is a nonzero ideal. Therefore \(1 = (1-e)^2 + 2e - e^2 \in \text{PI}(J)\), and 1 is a PI-element, hence \(J = J_1\) is PI. A contradiction.

Later, we will need to distinguish primitizers that satisfy a given PI in every homotope from primitizers which do not. We will say that an inner \(K\)-V pair is homotope-PI in \(V\) if there is an essential Jordan polynomial \(f \in FJ[X]\) which is an identity of \(K\) viewed as a subalgebra of every homotope \(V^{\sigma(e)}\), \(e \in V^{-}\), i.e., \(f(V^{-}\sigma;K) = 0\). Homotope-PI inner ideals in Jordan triple systems and algebras are defined in a similar way. We first prove:

**Lemma 3.2.** Let \(J\) be a nondegenerate Jordan system (triple system, pair or algebra). if \(J\) has a nonzero inner ideal which is homotope-PI in \(J\), then \(\text{PI}(J) \neq 0\).

**Proof.** We prove it for triple systems, and the same proof works for pairs or algebras. Suppose that \(K \subseteq J\) is homotope-PI, and take \(0 \neq k \in K\) and \(a \in J\). The mapping \(P_k: J^{(P\sigma)} \to J^{(a)}\) is a homomorphism of algebras whose image is contained in \(P_k J \subseteq K\). Since \(K\) is PI in every homotope, \(P_k J\) is a PI subalgebra of \(J^{(a)}\), hence \(J^{(P\sigma)}/\text{Ker} P_k\) is PI. Now, \(\text{Ker} P_k = \text{Ker} k \subseteq \text{Ker} P_k a\), hence \(J^{(P\sigma)}/\text{Ker} P_k\) is a homomorphic image of \(J^{(P\sigma)}/\text{Ker} P_k\), and therefore it is PI. Thus \(P_k J\) is contained in \(\text{PI}(J)\), and \(\text{PI}(J) \neq 0\) by nondegeneracy.

We can apply these result to primitizers of Jordan pairs:

**Lemma 3.3.** Let \(V\) be a Jordan pair, and let \(K \subseteq V^\sigma\) be a weak primitizer with modulus \((a, b) \in V^\sigma \times V^{-}\). If \(K\) is PI in the homotope \(V^{\sigma(b)}\), then \(V^\sigma_b\) is simple and unital, and \(b\) is (von Neumann) regular. Moreover, if \(K \neq 0\) is homotope-PI in \(V\), then \(V^\sigma_b\) is PI, i.e. \(b \in \text{PI}(V^{-}\sigma)\).

**Proof.** By taking the opposite pair if necessary, we can assume that \(\sigma = +\). By 1.4, for the first assertions we can also assume that \(K\) is in fact a primitizer of \(V\). Then, by 0.13, \(K = K + \text{Ker} b/\text{Ker} b\) is a primitizer of \(V^+_b\), and it is PI. Thus, from 3.1 it follows that \(V^+_b\) is simple and unital, and in particular \(b\) is regular.

To prove the last assertion, notice that, by 3.2, \(\text{PI}(V) \neq 0\), hence the image \(\text{PI}(V^+_b)\) of \(\text{PI}(V^+_b)\) in \(V^+_b\) is an ideal, and it is nonzero, since otherwise \(Q_0 \text{PI}(V^+_b) = 0\) would imply \(b \in \text{Ann}_1(\text{PI}(V)) = 0\) by strong primeness.
of $V$ (McCrimmon, 1984, 1.7(i)). Therefore $\Pi(V^+) = V^+_b$ by simplicity of $V^+_b$, hence $V^+ = \Pi(V^+) + \text{Ker} \ b$. By regularity, we can extend $b$ to an idempotent $e = (e^+, e^-)$ with $e^- = b$. Since $e^- \in V^+ = \Pi(V^+) + \text{Ker} \ b$, we have $e^- = Q_e e^+ \in Q_e(\Pi(V^+) + \text{Ker} \ b) \subseteq \Pi(V^+)$, and $e^+ = Q_e e^- \in \Pi(V^+)$. Thus $e \in \Pi(V)$. \hfill \textbf{\textbullet}

We can now prove an analogue of 3.1 for Jordan pairs.

**Proposition 3.4.** Let $V = (V^+, V^-)$ be a Jordan pair with a nonzero weak primitizer $K \subseteq V^\sigma$. If $K$ is homotope-PI in $V$, then $V$ is simple of finite capacity, and satisfies a homotope-PI.

**Proof.** Taking the opposite pair if necessary, we can assume $\sigma = +$. Since $K$ is PI in every homotope, $K$ is PI in the homotope $V^+(b)$. Thus, by 3.3, $b$ is regular and $V^+_b$ is simple, PI, and unital, hence it has finite capacity by 0.17. In particular, $V$ is rationally primitive (0.23).

Now, we can extend $b$ to an idempotent $e = (e^+, e^-)$ with $e^- = b \in \Pi(V^+)$, and since $V^2(e) \cong V(V^+_b)$ (0.4), the pair $V^2(e)$ has finite capacity.

On the other hand, $V^2_0(e) \subseteq K$ by 2.1, hence the pair $V^2_0(e)$ is homotope-PI by 0.20. Moreover, since $V$ has nonzero socle, and $V^2_0(e)$, is a subquotient of $V$ (Loos and Neher, 1994, 1.12), either $V^2_0(e)$ also has nonzero socle or $V^2_0(e) = 0$ by Loos and Neher (1994, 2.7). In particular, if it is nonzero, it is primitive (Montaner, 1999, 4.4(i)), and since it is homotope-PI, it is simple with finite capacity by 0.24.

Thus, both $V^2_0(e)$ and $V^2 (e)$ have finite capacity, hence $V$ has finite capacity by Loos (1991b, Lemma 5(c)). In particular $V = \text{Soc}(V) = \Pi(V)$, and the rank of the elements of $V$ is bounded by the capacity of $V$, hence there is an essential polynomial which is an identity of all local algebras $V^{\sigma\sigma}$ by Montaner (1999, 4.7(i)), and $V$ is homotope-PI. \hfill \textbf{\textbullet}

**Corollary 3.5.** If a Jordan pair $V$ has a (weak) primitizer which is maximal-(weakly) modular and satisfies homotope-PI in $V$, then $V$ is simple and has finite capacity.

**Proof.** By taking the opposite pair if necessary, we can suppose that the weak primitizer is contained in $V^+$. Let $(a, b) \in V^+ \times V^-$ be a modulus for which $K$ is maximal-(weakly) modular.

If $K \neq 0$, the result follows from 3.4, so we can assume $K = 0$. By 3.3, $b$ is a regular element, and we can complete it to an idempotent $e = (e^+, e^-)$ with $e^- = b$. Then $V_0(e) \subseteq K = 0$, and $e$ is principal. Now, set $c = a - Q_e b$. The result follows.
Then, $Q_a V^+ \subseteq K=0$ (see the proof of 1.4), hence $0 = e = a - Q_a b$ since $V$ is primitive, hence nondegenerate (0.10). Now, let $a = a_2 + a_1$ be the Peirce decomposition of $a$ relative to $e$. Then $B_{a,b} = e^+ \in K = 0$ by (weak) modularity, hence $B_{a,b} = e^+ - a_2 = 0$, and $Q_a b = Q_a a = a_2 = e^-$. Therefore $(a, b)$ is an idempotent, and we can take $e = (a, b)$. Moreover, $V_1(e)$ is an $(a, b)$-modular inner ideal (2.7), and $K = 0 \subseteq V_1(e)$. Thus, $V_1(e) = 0$ by maximality of $K$. Then $V = V_2(e)$, and any inner ideal is $(a, b)$-modular, hence $V$ does not have proper nonzero inner ideals by maximality of $K = 0$. Therefore, $V$ is a division pair (Loos, 1975, Proposition 10.4), and has capacity 1.

\[ \square \]

4. PROPERNESS

To check maximality of a maximal-modular inner ideal $K$ we will need a criterion for properness of inner ideals $M \supseteq K$. We will show that 0.7 extends to such an $M$ so that $M$ is proper if and only if it does not contain a modulus for $K$. The proof of that fact is rather technical, and makes use of the following description of the ideal generated by a 0-Peirce subspace.

**Lemma 4.1.** Let $V$ be a Jordan pair, and $e = (e^+, e^-)$ be an idempotent of $V$. Consider the Peirce decomposition $V = V_2(e) + V_1(e) + V_0(e)$ of $V$ relative to $e$, and set

\[
\begin{align*}
U_0^+(e) &= \{V_0^+(e), V_1^+(e), V_2^+(e)\}, \\
U_1^+(e) &= \{U_1^+(e), V_1^-(e), V_2^+(e)\}, \\
W_2^+(e) &= Q V_2^+(e), \text{ and } V_0^-(e).
\end{align*}
\]

Then the ideal $I = id_1(V_0^+(e))$ generated by $V_0^+(e)$ has

\[
I^+ = V_0^+(e) + U_0^+(e) + (U_1^+(e) + W_2^+(e)), \quad \text{and} \quad I^- = Q_V I^+.
\]

**Proof.** Set $U^+ = V_0^+ + U_1^+ + U_2^+$ and $W^+ = U^+ + W_2^+$. Clearly $W^+ \subseteq I^+$ and $Q_V I^+ \subseteq I^-$. Let us prove the reverse containments.

First, if $x_1 \in V_1^+$, $x_0 \in V_0^+$, and $a \in V^+$, from JP14 we get $\{x_1, e^+, \{e^+, x_0, a\}\} - \{e^+, x_0, \{x_1, e^+, a\}\} = \{x_1, e^+, e^+, x_0, a\} - \{e^+, \{e^+, x_1, x_0, a\}\}$, hence $\{x_1, x_0, a\} = \{e^+, e^+, x_1, x_0, a\}$ since we have $x_1 = \{x_1, e^+, e^+, \{x_1, e^+, x_0, a\}\} = 0$, and $\{e^+, x_0, a\} = 0$. In particular, $D_{V_1^+, V_0} V_0^+ \subseteq \{V_2^+, V_1^-, V_0^+\} = U_1^+$. 

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Then \( D_{V^+,V^-}V_0^+ = (D_{V_1^+,V_0^+} V_0^+ + D_{V_1^-,V_0^+} V_0^+) + (D_{V_2^+,V_1^+} V_0^+ + D_{V_2^-,V_1^+} V_0^+ ) \subseteq V_0^+ + U_1^+ + D_{V_1^+,V_0^-}V_0^+ = V_0^+ + U_1^+ \subseteq U^+ \).

Now, for any \( X \subseteq V^+ \), JP15 gives \( D_{V^+,V^-}D_{V_0^+V_1^-}X \subseteq D_{V_0^+V_1^-}D_{V^+,V^-}X \). In particular, if \( X = V_0^+ \), we get \( D_{V^+,V^-}U_1^+ \subseteq D_{V_1^+,V_0^+} V_0^+ + D_{V_1^-,V_0^+} V_0^+ \subseteq D_{V_1^+,V_0^+} (V_0^+ + U_1^+) + (V_0^+ + U_1^+) \) (since \( D_{V_1^+,V_0^+} V_0^+ \subseteq V_0^+ + U_1^+ = U_1^+ + V_0^+ + U_1^+ = U^+ \)).

Next, setting \( X = U_1^+ \) yields \( D_{V^+,V^-}U_2^+ \subseteq D_{V_2^+,V_1^+} V_2^+ + U^+ \subseteq D_{V_2^+,V_1^+} V_2^+ + U^+ = U^+ \) (since \( D_{V_2^+,V_1^+} U_2^+ \subseteq Q_{V_2^+} V_2^+ = 0 \), and \( D_{V_1^+,V_0^+} U_1^+ + V_0^+ + U_1^+ \subseteq U^+ \)).

Therefore, we have proved
\[ D_{V^+,V^-}U^+ \subseteq U^+. \] (i.1)

Now, writing JP9 and JP13 in the forms:
\[ Q_{x,z}Q_y = D_{x,y}D_{z,y} - D_{x,z}Q_y, \]
\[ Q_{x,y,z} = D_{x,y}D_{z,x} - D_{Q_{y,z}}, \]
we obtain from (i.1)
\[ Q_{V^+,V^-}U^+ \subseteq U^+, \] (ii.1)
and
\[ Q_{V^-,V^-}U^+ \subseteq U^+. \] (ii.2)

Also, writing JP10 in the form:
\[ D_{x,y}Q_z = Q_{x,z}D_{y,x} - Q_{D_{y,z}}, \]
we get \( D_{V^+,V^-}W_2^+ \subseteq D_{V_1^+,V_-}V_0^+ \subseteq (Q_{V^+,V_-}D_{V_1^+,V_-}V_0^+ + Q_{V^+,V_-}V_0^+ + Q_{V^-,V_-}V_0^+ + Q_{V^-,V_-}V_0^+ + Q_{V^-,V_-}V_0^+) \subseteq Q_{V^-,V_-}Q_{V^-,V^-}U^+ \) (by (i.1)) \( \subseteq U^+ \) (by (ii.1)).

Therefore, we have
\[ D_{V^+,V^-}W^+ \subseteq U^+ \subseteq W^+, \] (i)
and, using JP9 and JP13 as before
\[ Q_{V^+,V^-}Q_{V^-}W^+ + Q_{V^-}Q_{V^-}W^+ \subseteq W^+, \] (ii.3)

Next, writing JP5 in the form:
\[ Q_{x}D_{y,z} = Q_{x,(x,y,z)} + Q_{z,y} - Q_{x,y}D_{y,z}, \]
we get for any \( X \subseteq V^+ \), \( Q_{V^+} \cdot D_{V^+} \cdot X \subseteq Q_{V^+} \cdot X + Q_{V^+} \cdot D_{V^+} \cdot X \).

Thus, taking \( X = V^+_0 + U^+_1 \), we have \( Q_{V^+} \cdot (U^+_1 + U^+_1) \subseteq Q_{V^+} \cdot D_{V^+} \cdot (V^+_0 + U^+_1) \), \( Q_{V^+} \cdot D_{V^+} \cdot (V^+_0 + U^+_1) \subseteq Q_{V^+} \cdot U^+ \).

On the other hand, using the decomposition \( V = V_0 + V_1 + V_2 \), we obtain

\[
Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 + Q_{V^+} \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+ \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+ \cdot V^+_0 + Q_{V^+} \cdot V^+ \cdot V^+_0 \subseteq U^+ + Q_{V^+} \cdot V^+_0 + W^+_2 \quad (\text{by (ii.1)})
\]

and the containments \( Q_{V^+} \cdot Q_{V^+} \cdot V^+_0 \subseteq V^+_0 \subseteq U^+ \).

Now, if \( x_1 \in V^+_1 \), \( x_0 \in V^+_0 \), and \( y_0 \in V^+_0 \), from JP20 we get

\[
Q_{V^+} \cdot Q_{V^-} \cdot Q_{x_0} = Q_{V^+} \cdot Q_{x_0} = Q_{V^+} \cdot Q_{x_0} \cdot x_0 \quad (\text{by (ii.1)})
\]

Thus, \( Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq Q_{V^+} \cdot V^+_0 \), and \( Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 = Q_{V^+} \cdot V^+_0 \).

Therefore we get \( Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq W^+_2 \), hence

\[
Q_{V^+} \cdot Q_{V^-} \cdot U^+ \subseteq W^+. \quad \text{(ii.4)}
\]

Next, \( Q_{V^+} \cdot Q_{V^-} \cdot W^+_2 \subseteq Q_{V^+} \cdot Q_{V^-} \cdot W^+_2 + Q_{V^+} \cdot Q_{V^-} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 \)

\[
W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 \subseteq U^+ + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2.
\]

Now, from JP20 we get

\[
Q_{V^+} \cdot Q_{V^+} \cdot W^+_2 = (Q_{V^+} \cdot Q_{V^-} \cdot Q_{V^+}) \cdot Q_{V^+} \cdot W^+_2 \subseteq (Q_{V^+} \cdot Q_{V^-} \cdot Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot Q_{V^-} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2)
\]

\[
W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 \subseteq U^+ + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2 + Q_{V^+} \cdot W^+_2.
\]

(by \( Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq V^+_0 \) (ii.3) and (ii.4)).

On the other hand, also applying JP20, we have

\[
Q_{V^+} \cdot Q_{V^-} \cdot W^+_2 = Q_{V^+} \cdot (Q_{V^-} \cdot Q_{V^-} \cdot Q_{V^+}) \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 + Q_{V^+} \cdot Q_{V^-} \cdot V^+_0
\]

\[
+ Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq (Q_{V^+} \cdot Q_{V^-} \cdot V^+_0)^2 + Q_{V^+} \cdot Q_{V^-} \cdot V^+_0 \subseteq W^+.
\]
Therefore, we obtain \( Q_{V^-}Q_{V}W^+ \subseteq W^+ \), which, together with (ii.4) gives
\[
Q_{V^-}Q_{V^-}W^+ \subseteq W^+.
\]

(ii)

Now, JP10 yields \( D_{V^-,V^-}Q_{V^-}W^+ \subseteq Q_{V^-}D_{V^-,V^-}W^+ + Q_{V^-}W^+ \subseteq Q_{V^-}W^+ \) by (i), i.e.
\[
D_{V^-,V^-}Q_{V^-}W^+ \subseteq Q_{V^-}W^+.
\]

(iii)

Consider now the triple system \( T = T(V) \). Clearly \( K = V^0_0 \oplus 0 \) is an inner ideal of \( T \). Then, by Montaner (to appear in J. Algebra, 4.4) the ideal generated by \( K \) in \( T \) is \( \mathcal{M}(T)_I \), where \( \mathcal{M}(T) \) is the (unital) multiplication algebra of \( T \), generated by the identity and all the operators \( P_{x^+y^-z}, L_{x^+y^-z}, \) for \( x^+ \oplus y^+ \oplus z \in T \). Then we have \( \mathcal{M}(T)_I = I^+ \oplus I^- \).

On the other hand (i), (ii) and (iii) show that \( \mathcal{M}(T)_I (W^+ + Q_{V^-}W^+) \subseteq W^+ + Q_{V^-}W^+ \), and since \( K \subseteq W^+ + Q_{V^-}W^+ \subseteq I^+ \oplus I^- \), we have \( I^+ \oplus I^- \subseteq W^+ + Q_{V^-}W^+ \), hence \( I^+ = W^+ \), and \( I^- = Q_{V^-}W^+ \).

4.2. Let \( J \) be a Jordan algebra, and \( K \) be an inner ideal of \( J \), recall (McCrimmon, 1982) that the linear absorber of \( K \) is defined as \( l(K) = \{ z \in K \mid z \circ J \subseteq K \} \), and the quadratic absorber is \( q(K) = \{ z \in K \mid V_{J,J} + U_{J,J} \subseteq K \} \). It is proved in McCrimmon (1982, Sec. 2) that \( l(K) \) and \( q(K) \) are inner ideals of \( J \) and ideals of \( K \), and they satisfy
\[
q(K) \subseteq l(K) \quad \text{and} \quad U_{l[K]}K + l(K)^2 \subseteq q(K).
\]

Moreover, the ideal \( \text{id}(q(K)) \) generated by the quadratic absorber is nil modulo \( q(K) \) (McCrimmon, 1982, 3.7).

If \( V = (V^+,V^-) \) is a Jordan pair, \( K \subseteq V^0 \) is an inner ideal, and \( b \in V_0^\sigma \), we define the linear \( b \)-absorber \( l(b;K) \) and the quadratic \( b \)-absorber \( q(b;K) \) as the linear and quadratic absorber of \( K \) in the homotope \( V^0(b) \), respectively. From the properties of the absorbers we obtain \( q(b;K) \subseteq l(b;K) \), \( Q_{b;K}Q_{b;K} + Q_{b;K}b \subseteq q(b;K) \), and the ideal \( \text{id}(q(b;K)) \) generated by \( q(b;K) \) in the homotope \( V^0(b) \) is \( b \)-nil modulo \( q(b;K) \): for any \( x \in \text{id}(q(b;K)) \) there is \( n \geq 1 \) such that \( x_n(b) \in q(b;K) \).

**Proposition 4.3.** Let \( V = (V^+,V^-) \) be a Jordan pair, \( K \subseteq V^0 \) be an inner ideal, and suppose that \( K \) is maximal-(weakly) \( (a,b) \)-modular for some \( (a,b) \in V^0 \times V_0^\sigma \). If \( L \subseteq V^0 \) is an inner ideal with \( K \subseteq L \), then \( L = V^0 \) if and only if \( a \in L \).
Proof. Clearly $L = V^n$ implies $a \in L$. For the reciprocal, suppose that $a \in L$.

Passing to the quotient by Core($K$) we can assume that $K$ is a primitizer. Set $J = V^n(b)$ and $J = V^n$. We denote with bars the projections in $J$. Taking the opposite pair if necessary, we can assume $\sigma = +$.

Take $I = \text{id}(q(b; L)) + \text{Ker} b$. Suppose first that $I \neq 0$, then there exists a nonzero ideal $N = (N^+, N^-)$ of $V$ which is $b$-nil modulo $I$ by 0.14. Since $K$ is a primitizer, we have $K + N^+ = V^+$, hence $a = k + a_1$ for some $k \in K$ and $a_1 \in N^-$. Then, $(a_1, b)$ is a (weak) modulus for $K$ by 0.8. By $b$-nilness of $N$ mod $I$, there exists $n \geq 1$ such that $a_2 = a_1^{(n,b)} \in I$, therefore $a_3 = a_1^{(n+2,b)} = Q_a Q_{a_2} \in Q_{V^+}(\text{id}(q(b; L)) + \text{Ker} b) \subseteq \text{id}(q(b; L))$. Now, by nilness of $\text{id}(q(b; L)) \mod q(b; K)$ there exists $m \geq 1$ such that $c = a_3^{(m,b)} \in q(b; L)$. Note that $(c, b) = (a_1^{(n+2m,b)}, b)$ is a (weak) modulus for $K$ by 0.8.

Now, if $x \in V^+$, $B_{x,b}x \in K \subseteq L$, but $B_{x,b}x = x - \{c, b, x\} + Q_a Q_{b}x$, and $\{-c, b, x\} + Q_a Q_{b}x \in L$ since $c \in q(b; L)$. Therefore $x \in L$. Then $L = V^+$ and we are done.

Thus we can assume $I = 0$, hence $q(b; L) = 0$. On the other hand, if $k \in K$ and $x \in V^+$, we have $\{k, b, x\} - \{k, P_{b}x, a\} \in K$ by (weak) modularity, but since $a \in L$, $\{k, P_{b}x, a\} \in Q_{V^+} \subseteq L$, and we get $\{k, b, x\} \in L$. This proves the containment $K \subseteq \langle b, L \rangle$. Then we have $Q_{[b,L]}Q_{b}L \subseteq Q_{[b,L]}Q_{b}L \subseteq q(b; L) = 0$, so in particular, $U_{K}U_{K} = 0$ in $J$, and $K$ is PI in $J$. Then, by 3.3, $b$ is regular in $V$. Let us see that $b$ extends to an idempotent $e = (e^+; e^-)$, with $e^- = b$ and $e^+ \in L$. First extend $b$ to an idempotent $f = (f^+; f^-)$, $f^- = b$, and take $a = a_2 + a_1 + a_0$ the Peirce decomposition relative to $f$. By 2.1, $V_0^+(f) \subseteq K$, hence we can suppose $a_0 = 0$ by 0.8. Now setting $c = f^+ + a_1 + Q_{a} f^-$, we have $c - a = f^+ - a_2 + Q_{a} f^- \equiv f^+ - a_2 \mod K$ since $Q_{a} f^- \in V_0^+(f) \subseteq L$ by 2.1(a), and a straightforward computation shows that $f^+ - a_2 = B_{a,b} f^+ + (a - Q_{b} b) \in L$ (by (weak) modularity of $K$, and $a \in L$), hence $c \in L$. A routine check shows that $(c, b)$ is an idempotent, so it suffices to take $e^+ = c$.

Thus, $V_2^+(e) \subseteq L$ since $e^+ \in L$, and $V_0^+(e) \subseteq K \subseteq L$ by 2.1. Then, if $I = (I^+, I^-)$ is the ideal generated by $V_0^+(e)$ in $V$, we have $I^+ \subseteq L$ by Lemma 4.1. If $I \neq 0$, $V^+ = I^+ + K \subseteq L$, and we are done. Therefore $I = 0$, hence $V_0^+(e) = 0$, and $e$ is principal.

Now, $Q_{b} Q_{a} Q_{b} L \subseteq 0$ and $V_2^+ (e) \subseteq L$ imply $0 = Q_{c} Q_{e} Q_{a} Q_{e} Q_{c} V^+ = Q_{c} Q_{e} K = 0$ by nondegeneracy. Therefore $K \subseteq V_1^+(e) + V_0^+(e) = V_1^+(e)$. On the other hand, $V_1^+(e)$ is an inner ideal since $e$ is complete. Moreover, $B_{a,b} V^+ \subseteq K \subseteq V_1^+(e)$, and for all $x \in V_1^+(e)$, $y \in V^+$, $\{x, b, y\} - \{x, Q_{b} y, a\} \in \{V_1^+(e), V_2^+(e), V^+\} \subseteq V_1^+(e)$, hence $V_1^+(e)$ is weakly $(a,b)$-modular, and if $K$ is $(a,b)$-modular, we have
a - Q, b \in K \subseteq V_1^+(e). But also \{V_1^+(e), b, a\} \subseteq \{V_1^+(e), e^-, V^+\} \subseteq V_1^+(e) (since V_0(e) = 0), hence \{V_1^+(e)\} is also \(a, b\)-modular. Therefore \(K = V_1^+(e)\) by maximality, and \(V^+ = V_2^+(e) + V_1^+(e) \subseteq L\).

The triple version of 4.3 is

**Proposition 4.4.** Let \(J\) be a Jordan triple system, \(K \subseteq J\) be an inner ideal of \(J\), and suppose that \(K\) is maximal (weakly) \((a, b)\)-modular for some \(a, b \in J\). If \(L\) is an inner ideal of \(J\) with \(K \subseteq L\), then \(L = J\) if and only if \(a \in L\).

**Proof.** Consider the pair \(T(J)\) and use the notation of 0.12: for \(S \subseteq J, \sigma = \pm\), we write \(S^\sigma\) for \(S\) viewed as a subset of \(V(J)^\sigma\). Then, by 0.12(b), \(K^\sigma\) is maximal (weakly) \((a^\sigma, b^\sigma)\)-modular, and \(K^+ \subseteq L^+\), so the result follows from the pair case 4.3.

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**5. CONSTRUCTION OF AN IDEAL OF POLYNOMIALS**

The study (Anquela et al., 1993a) of maximal-modular inner ideals in quadratic Jordan algebras required the construction of polynomial ideals \(G(X)\) that behaved well with respect to inner ideals of special Jordan algebras in the sense that, for any special Jordan algebra \(J\) with associative envelope \(R\), and any inner ideal \(K\) of \(J\), \{\(G(K), J, J, J, K\)\} \(\subseteq K\). More precisely (Anquela et al., 1995, Lemma 2.3):

**Theorem 5.1.** There exists a nonzero, linearization invariant \(T\)-ideal \(G(X)\) of the free special Jordan algebra \(SJ[X]\) such that:

(a) \(G(X)\) consists of hearty pentad eaters and contains Clifford identities.

(b) For any special Jordan algebra \(J\), any associative algebra with involution \((R, \ast)\) such that \(J \subseteq H(R, \ast)\), and any inner ideal \(K\) of \(J\),

\[\{G(K), j, j, j, K\} \subseteq K\]

where the pentads are taken in \(R\).

**5.2.** Recall that if \(T(X) \subseteq SJ(X)\), and \(Y \subseteq SJ[X]\), we denote by \(T(Y; X)\) the subset of \(SJ[X]\) formed by the polynomials \(f(y; x_1, \ldots, x_n)\) for \(f(x_1, \ldots, x_n) \in T(X)\) and \(y \in Y\). It was proved in Anquela and Cortès
(1996, 2.2) that, for all odd $n \geq 5$, there exists a nonzero linearization invariant $\mathcal{F}$-ideal $\mathcal{H}_n(X)$ of $\text{SJ}[X]$ such that $\mathcal{H}_n(\text{SJ}[X]; X) \subseteq \mathcal{H}_n(X)$.

Now, for any $x, y, z \in X$, define

$$q(x, y, z) = U_z^2 \left[ (U_x^2 x) o_{(z)} x - U_y^2 (x^{(2, z)}) - U_z^2 (y^{(2, z)}) \right] \in \text{SJ}[X],$$

and $r(x, y, z) = q(x, y, z)^2 \in \text{SJ}[X]$.

Denote by $\mathcal{G}(X)$ for all odd $n$, the $n$th derived ideal of $\mathcal{G}(X)$:

$$\mathcal{G}^1(X) = \mathcal{G}(X),$$

and inductively, $\mathcal{G}^{n+2}(X) = U_{\mathcal{G}(X)} \mathcal{G}^n(X)$, and take

$$S(X) = \{ r(f, g, h) \mid f \in \mathcal{G}(\mathcal{H}_{23}(X)), g, h \in \mathcal{H}_{19}(X) \} \subseteq \text{SJ}[X].$$

It has been proved in Anquela and Cortés (1996, 2.3) that, for any special Jordan triple system $T$ which is a subsystem of $H(R, *)$ for an associative triple system with involution $(R, *)$, and any inner ideal $K$ of $T$,

$$\{ S(T; K) \text{TTTK} \} \subseteq K.$$

Moreover, $S(X)$ is nonzero, indeed $S(H_3(\Phi)) \neq 0$.

\textbf{5.3. We now give a simpler construction of polynomials satisfying these properties. As above, consider a special triple system $T$ which is a subsystem of $H(R, *)$ for an associative system $R$ with involution $*$, an inner ideal $K$ of $R$, and fix $b \in T$. We denote by $\equiv$ the congruence modulo $K$: $x \equiv y$, for $x, y \in T$ if $x - y \in K$.

Take $v \in \mathcal{G}(b; K)$, and set $a = P_x b \in \mathcal{G}(b; K)$ (recall that $\mathcal{G}(X)$ is an ideal, so in particular, it is closed under squaring). As in (D’Amour, 1991, 4.2; Anquela and Cortés, 1996, 2.3), for any $x, t \in T$ we denote $L_x(t) = \{ abxb \}$, and $L_x^2(t) = \{ bxbat \}$, so in particular $L_a(T) = \{ a, P_x x, T \} \subseteq T$, $L_a(K) = \{ a, P_b x, K \} \subseteq K$ (since $a \in K$), and $L_a^2(T) = \{ P_b x, a, T \} \subseteq T$. Then the following congruences are proved in Anquela and Cortés (1996, 2.3), for $k \in K$, $x_1, x_2, z \in T$:

$$\{ kx_1 L_z(x_2)ba \} \equiv \{ kL_z^2(x_1)x_2ba \} \equiv \{ L_z(k)x_1x_2ba \},$$

from which it follows, for any $x, y \in T$:

$$\{ kx_1 [L_x, L_y](x_2)ba \} \equiv 0.$$
Next, we denote, for any \( x \in T, r \in R, E_x(r) = xhabr \) (note that this is just the left multiplication by \( x \) in the homotope \( R^{P_{a(b)}} \)). Now, take \( y_1, \ldots, y_n \in T \), and set \( E = E_{y_1} \cdots E_{y_n} \). We claim that the following congruence holds for any \( k \in K, x_1, x_2, z \in T \):

\[
\{kx_1x_2bE_zE(a)\} \cong \{kx_1L_z(x_2)bE(a)\}. \tag{3}
\]

To prove that, we first note the containment (in \( R \)):

\[
TbTbE(a) \subseteq TbTbTba. \tag{\ast}
\]

Indeed, computing in the \( b \)-homotopes \( R^{P_{a(b)}} \), with product \( r/b/a \equiv rbsa \), and \( J = T^{P_{a(b)}} \) (with unitization \( \bar{b} \)), we have

\[
\begin{align*}
TbTbTbE(a) & \subseteq TbTb(Tbab)^n a \subseteq \bar{b}^n (J \cdot (b) J \cdot (b) (J \cdot (b) a)^{n,b} \cdot (b) a) \\
& \subseteq \bar{b}^n (J \cdot (b) J \cdot (b) (J \cdot (b) H^5(J))^{(n,b)} \cdot (b) a) \text{ (since } a \in \mathcal{U}(b; K)) \\
& \subseteq H^5(J) \text{ by } 5.1 (\text{a}) \\
& = \bar{b}^n (J \cdot (b) J \cdot (b) (J \cdot (b) H^5(J))^{(n-1,b)} \cdot (b) a) \\
& \subseteq J \cdot (b) J \cdot (b) (J \cdot (b) H^5(J))^{(n-1,b)} \cdot (b) a \text{ (since } H^5(J) \text{ eats associative pentads)} \\
& \subseteq \cdots \subseteq J \cdot (b) J \cdot (b) a \text{ (inductively)} = TbTbTba.
\end{align*}
\]

Returning to the proof of (3), we have

\[
\begin{align*}
\{kx_1x_2bE_zE(a)\} & = \{kx_1x_2bzbabE(a)\} \\
& = \{kx_1L_z(x_2)bE(a)\} - \{kx_1abzbx_2bE(a)\},
\end{align*}
\]

and

\[
\begin{align*}
\{kx_1abzbx_2bE(a)\} & \subseteq \{KTabTbTbE(a)\} \subseteq \{KTabTbTbTba\} \text{ (by } \ast). \tag{\ast}
\end{align*}
\]

So (3) will follow if we can show that \( \{KTabTbTbTba\} \subseteq K \).

Now we have

\[
\begin{align*}
\{KTabTbTbTba\} & \subseteq \{\{KTa\}bTbTbTba\} + \{aTkTbTbTba\} \subseteq \{\{KTa\}bTbTbTba\} + \{\{P_aT\}bTbTbK\} \\
& \subseteq \{KbTbTbTba\} + \{KT\{KbTbTbTba\}\} + \{\{P_aT\}bTbTbK\} \\
& \text{ (since } a \in K). \tag{\ast}
\end{align*}
\]
Denote the \( n \)-tads in \( R(b) \) with the superscript \((b)\), and write \( K^{(b)} \) for \( K \) viewed as an inner ideal of \( J = T^{(b)} \). With that notation \( \{KbTB\{Ta\} = \{K^{(b)}(\mathcal{J}J\mathcal{J}a)\}^{(b)} \) (since \( a \in \mathcal{G}(b; K) = \mathcal{G}(K^{(b)}) \) in \( K \) (by property 5.1(b)). Hence also \( \{KT\{KbTB\{Ta\}\} \subseteq \{K, T, K\} \subseteq K \). Next, we have \( \{(P_\alpha T)bTB\{Ta\}Tb\} \subseteq \{3\{(b; K)\}P_bK\} \) (since \( a = P_\alpha b \) with \( a \in \mathcal{G}(b; K) \) gives \( P_\alpha T = P_\alpha P_b P_\alpha T \subseteq P_\alpha \mathcal{G}(b; K)P_b K \) \( \subseteq \mathcal{G}(b; K)P_b K = U^{(b)}_{(b; K)} \) \( K \subseteq \mathcal{G}(b; K) \), since \( \mathcal{G}(X) \) is an ideal) \( \subseteq K \) as before. And this proves (3).

Next, take \( y_1, \ldots, y_m, z_1, \ldots, z_m \in T \), and set \( E = E_{y_1}, \ldots, E_{y_m}, E' = E_{z_1}, \ldots, E_{z_m} \); we have, for all \( k \in K \), and \( x, y, x_1, x_2 \in T \):

\[
\{kx_1 x_2 bE'[E_x, E_y]E(a)\} \equiv 0. \tag{4}
\]

Indeed, setting \( L = L_{y_1} \cdots L_{y_m}, \ L^* = L_{y_1}^* \cdots L_{y_m}^* \) and \( L' = L_{z_1} \cdots L_{z_m} \), we have \( \{kx_1 x_2 x_3 bE'[E_x, E_y]E(a)\} \equiv \{kx_1 L[L_{y_1}, L_{y_2} L_{y_3} b] \} \) (by repeated use of (3)) \( \equiv \{kL' x_1 L_{y_1}, L_{y_2} L_{y_3} b] \} \) (by repeated use of (1)) \( \equiv 0 \) (by (2)).

Now set

\[
p = P_\alpha P_\beta (c, y, z)^{P(\beta)} (y, x, z)^{P(\alpha)} \cdot \alpha \cdot (x, y)^{P(\alpha)} (y, z)^{P(\beta)} \cdot \beta \cdot \alpha \cdot a
\]

(where, again, superscripts \( P(\alpha) \) denote products in the \( P(\alpha) \)-homotopes). Then \( p = E_{[x, y, z]}(E_x, E_y, E_z) \) (a), hence from (4) we get

\[
\{KTTBPb\} \subseteq K.
\]

Now, choose \( v \in \mathcal{G}(b; K) \cap \mathcal{H}_{(b, K)} \subseteq \mathcal{H}_{(b, T)} \) to define \( a = P_\alpha b \) (then \( p \) has the form \( p = P_\alpha c \) for some \( c \), and take \( x_1, x_2, x_3 \in T \), \( k \in K \).

We have, computing as in Anquela and Cortés (1996, 2.3), \( \{kx_1 x_2 x_3 P_\beta b\} = \{kx_1 \{x_2 x_3 P_\beta b\}\} - \{kx_1 P_\beta \{x_2 x_3 b\}\} + P_\beta \{kx_1 x_2 x_3 b\} \in \{KTTBP\} + P_\beta \{TTTTT\} \subseteq K + P_\beta P_\alpha \{TTTTT\} \) (since \( v \in \mathcal{H}_{(b, T)} \) is a hearty pentad-eater) \( \subseteq K + P_\beta P_\alpha \{TTTT\} \) (again since \( v \) is a hearty pentad-eater) \( \subseteq K + P_\alpha P_\beta P_\alpha T \subseteq K \). Therefore we have:

\[
\{KTTTTBPb\} \subseteq K.
\]

Thus defining

\[
h(x, y, z, t) = U^\beta (\mathcal{G}_{x,y}^\beta z),
\]

where \( \mathcal{G}_{x,y}^\beta = \{x, y, z\}^{(w)} - \{y, x, z\}^{(w)} \), we have, for \( x, y, z \in T \), and \( v \in \mathcal{G}(b; K) \cap \mathcal{H}_{(b, K)} \), \( h(b; x, y, z, v) = p \), and setting \( f(x, y, z, t) = h(x, y, z, t^2) \in \mathcal{S}(x, y, z, t) \), we get \( f(b; x, y, z, v) = P_\beta b \).
Then, the set
\[ F(X) = \{ f(x,y,z) \mid g \in \mathcal{G}(\mathcal{H}_5(X)), x,y,z \in X \} \subseteq SJ[X] \]
has
\[ \{ KTTTF(T; K) \} \subseteq K, \]
for any special Jordan triple system \( T \) which is a subsystem of \( H(R, \ast) \) for an associative triple system with involution \((R, \ast)\), and any inner ideal \( K \) of \( T \). Moreover, \( F(X) \) contains Clifford identities. Indeed, \( \mathcal{H}(H_3(\Phi)) = H_3(\Phi) \) (Anquela and Cortés, 1996, 2.2), and \( \mathcal{G}(H_3(\Phi)) = H_3(\Phi) \) since \( \mathcal{G}(X) \) contains Clifford identities (5.1 (a)). Thus, \( \mathcal{G}(\mathcal{H}(H_3(\Phi))) = H_3(\Phi) \), so there is a \( g \in \mathcal{G}(\mathcal{H}_5(X)) \) which takes the value \( 1 \in H_3(\Phi) \) under a suitable evaluation. Thus, taking \( x = u_{12} + e_{23}, y = u_{13} = e_{13} + e_{31} \) and \( z = e_{33} \) for matrix units \( e_j \in H_3(\Phi) \), and the previous evaluation of \( g \) (noticing that since \( X \) is infinite we can choose the variables in \( g \) different from \( x, y, z \), the polynomial \( f \) takes the value \( f(u_{12}, u_{13}, e_{33}, 1) = h(u_{12}, u_{13}, e_{33}, 1)^2 = (\{[u_{12}, u_{13}] \})^2 = (e_{23} + e_{32})^2 = e_{22} + e_{33} \). Thus \( F(H_3(\Phi)) \neq 0 \), and \( F(X) \) contains Clifford polynomials.

**Theorem 5.4.** There exists a nonzero linearization invariant \( \mathcal{F} \)-ideal \( \mathcal{F}(X) \) of \( SJ[X] \) such that for any special Jordan triple system \( T \), any associative triple system with involution \((R, \ast)\) such that \( T \subseteq H(R, \ast) \), and any inner ideal \( K \) of \( T \),
\[ \{ \mathcal{F}(T; K)T T T K \} \subseteq K. \]
where the pentads are taken in \( R \). Moreover, \( \mathcal{F}(X) \) contains clifford identities.

**Proof.** Define \( \mathcal{F}(X) \) as the set of polynomials \( f(x_1, \ldots, x_n) \in \mathcal{H}_5(X) \) such that for any special Jordan triple system \( T \), any associative triple system with involution \((R, \ast)\) such that \( T \subseteq H(R, \ast) \) is a subtriple, and any inner ideal \( K \subseteq T \), \( \{ f(T; K)T T T K \} \subseteq K. \)

Let us see that \( \mathcal{F}(X) \) is an ideal of \( SJ[X] \). Take \( f \in \mathcal{F}(X), p \in SJ[X] \). Note that, since \( \mathcal{H}_5(X) \) is an ideal, \( f \circ p, f^2, U_{ij}f, \) and \( U_{ij}p \) are all in \( \mathcal{H}_5(X) \). Let \( K \subseteq T \subseteq H(R, \ast) \) be as before, and choose \( b \in T, c \in f(b; K), \) and \( k \in p(b; K) \). Then, for any \( k' \in K, t_1, t_2, t_3 \in T \), we have:

- \( \{ \{ ebk \}_{t_1 t_2 t_3 k'} = \{ c \{ b k t_1 \}_{t_2 t_3 k'} \} - \{ c t_1 \{ b k t_2 \}_{t_3 k'} \} - \{ c t_1 t_2 \{ b k t_3 \} k' \} - \{ c t_1 t_2 t_3 \{ k b k' \} \} + \{ c t_1 t_2 t_3 k' \} \} b k \in \{ c T T K \} + \{ c T T T K \}TK \subseteq K. \) Thus \( f \circ p \in \mathcal{F}(X) \).
identities.

\[ \{(P, b) t_1 t_2 k \} = \{ cb t_1 t_2 k \} \subseteq \{ c T c T T T T T K \} \subseteq \{ c T T T K \} \] (since 
\[ c \in \mathcal{H}(T) \subseteq K. \] Thus \( f^X \in \mathcal{F}(X) \).

\[ \{(P, P b) t_1 t_2 k \} = \{ c(P, b) t_1 t_2 t_3 k \} \subseteq \{ c T c T T T T K \} \subseteq \{ c T T T K \} \] (since 
\[ c \in \mathcal{H}(T) \subseteq K. \] Thus \( U_{P b} \in \mathcal{F}(X) \).

\[ \{(P, P b) t_1 t_2 t_3 k \} = \{ k b c b t_1 t_2 t_3 k \} - \{ k b c b t_1 t_2 t_3 k \} + P_k \{ b \} \]
\[ \{ c b t_1 t_2 t_3 k \} \subseteq \{ k b c b t_1 t_2 t_3 k \} + P_k \{ b \} - \{ k b c b t_1 t_2 t_3 k \} + P_k \{ k b c b t_1 t_2 t_3 k \} \in \{ k b c b T T T K \} + P_k \{ T T T T T T T \} \] (since 
\[ c \in \mathcal{H}(T) \subseteq K. \] Thus \( \mathcal{F}(X) \subseteq \mathcal{F}(X) \).

\[ \{ T T T T T T T \} \subseteq T, \quad \{ k b c b t_1 t_2 t_3 k \} \subseteq \{ k b c b T T T T T T T \} \subseteq \{ k b c b T T T T T T T \} \subseteq K. \] Now, if \( x, y \in T \) and 
\[ x y c \in \{ k b c b T T T T T T T \} \subseteq K, \] hence \( \{ k b c b T T T T T T T \} \subseteq K. \]

Therefore \( \mathcal{F}(X) \) is an ideal of \( \mathbb{S} d[\mathbb{X}] \). That this is linearization invariant follows from the linearity of its definition, and it is obviously a \( \mathcal{F} \)-ideal. Moreover, it is nonzero since the family \( F(X) \subseteq \mathbb{S} d[\mathbb{X}] \) of 5.3 clearly has \( F(X) \subseteq \mathcal{F}(X) \), and \( F(X) \) is nonzero and contains clifford identities.

\[ \{ F^{-\sigma}; K \} = V^{-\sigma} V^\sigma V^{-\sigma} K \] \( \subseteq K. \)

This readily follows from 5.4 applied to \( T = T(V) \), which is a subsystem of \( H(R^+ \oplus R^-, *) \), for the polarized involution \( (r^+ \oplus r^-)^* = (r^+)^* \oplus (r^-)^* \), and to the inner ideal \( K \oplus 0 \) of \( T \).

6. NON HOMOTOPE PI MAXIMAL PRIMITIZERS

We next apply the polynomials constructed in the previous section in the study of primitizers of Jordan systems \( J \) which are not homotope-PI in \( J \).

**Lemma 6.1.** Let \( V \) be a Jordan pair, and suppose that \( K \subseteq V^\sigma \) is a weakly (\( a, b \))-modular inner ideal for some \( (a, b) \in V^\sigma \times V^{-\sigma} \). If \( c \equiv a^{(n, \ b)} \mod K_0, \) and \( c \equiv a^{(m, \ b)} \mod K_0 \) for some \( n, m \geq 1 \), then \( Q_n Q_0 d \equiv a^{(2n+m, \ b)} \mod K_0 \) is a \( b \)-modulus for \( K \).
Proposition 6.2. Let $V$ be a special Jordan pair, and let $R = (R^-, R^+)$ be an associative $*$-envelope of $V$. If $K \subseteq V^0$ is a maximal-weak primitizer for the modulus $(a, b) \in V^0 \times V^-$ which is not homotope-PI in $V$, and $L \subseteq V^0$ is a proper inner ideal containing $K$, then

$$L + LR^{-}R^+ \neq R^+.$$  

Proof. We can assume $\sigma = +$. For $X \subseteq V^0$ we write $X(V^{-}\delta V^{0})^{0} = X$, and inductively, $X(V^{-}\delta V^{0})^{k+1} = (X(V^{-}\delta V^{0})^{k})V^{-}\delta V^{0}$. (We define $(V^{0} V^{-})^{X}$ similarly.) Write also $XR^{-}R^{0} = X + XR^{-}R^{0}$. Suppose that $LR^{-}R^{+} = R^{+}$, then there exists $n \geq 0$ such that

$$a \in \sum_{i=0}^{n} L(V^{-}V^{+})^{i}. \tag{1}$$

Consider the ideal $\mathcal{F}(X)$ of 5.4. Since $K$ is not homotope-PI, $L$ is not homotope-PI, and since the ideal $\mathcal{F}^{3}(X) = U_{\mathcal{F}(X)}\mathcal{F}(X)$ does not contain any homotope-PI of $L$, hence there exists $c \in V^{-}$ with $Q_{\mathcal{F}(c)} LQ_{\mathcal{F}(c)} (c, L) = \mathcal{F}^{3}(c; L) \neq 0$. Therefore $0 \neq \mathcal{F}(c; L)c\mathcal{F}(c; L) \subseteq R^{+}$, and the ideal $I = \text{id}_{R}(\mathcal{F}(c; L)c\mathcal{F}(c; L))$ generated by that set in $R$ is nonzero. Moreover $0 \neq \mathcal{F}^{3}(c; L) \subseteq I \cap V^{+}$, hence $I \cap V^{+}$ is a nonzero ideal of $V$. By 1.3(c), $K_{0}$ is again a weak primitizer with weak modulus $(a, b)$. Thus, there exists a $b$-modulus $e \in I \cap V^{+}$ such that $a \equiv e \mod K_{0}$. Thus for some $m \geq 0,$

$$e \in \sum_{k=0}^{m} (V^{+}V^{-})^{k} \mathcal{F}(c; L)c\mathcal{F}(c; L)(V^{-}V^{+})^{l}. \tag{2}$$

Next, for all odd $N \geq 5$, let $\mathcal{H}_{N}(X)$ be a hearty $N$-tad eater ideal of SJT$[X]$, and denote by $\mathcal{H}_{N}(V)^{+}$ the $+$-part of its evaluation $\mathcal{H}_{N}(T(V))$ in $T(V) = V^{+} \oplus V^{-}$. Note that $\mathcal{H}_{N}(X)$ contains homotope polynomials (5.2), hence $\mathcal{H}_{N}(T(V)) \neq 0$, and $\mathcal{H}_{N}(V)^{-} \neq 0$. Thus, we can take a $b$-modulus $f_{N} \in \mathcal{H}_{N}(V)^{-}$ with $f_{N} \equiv a \mod K_{0}$ as before.
Then, if $f = f_{2(n+m)+3}$, we have by (1) and (2)

\[
abf be \in \sum_{k \leq n, l \leq m} L(V^- V^+)^k bfb(V^+ V^-)^l \mathcal{F}(c, L) c \mathcal{F}(c, L) R^- R^+
\]

\[
\subseteq \sum_{k \leq n, l \leq m} \{ L(V^- V^+)^k bfb(V^+ V^-)^l \mathcal{F}(c, L) c \mathcal{F}(c, L) R^- R^+
\]

\[
+ \mathcal{F}(c, L) R^- R^+
\]

\[
\subseteq \{ LV^- V^- \mathcal{F}(c, L) c \mathcal{F}(c, L) R^- R^+ + \mathcal{F}(c, L) R^- R^+ \subseteq
\]

(since $f$ eats embedded $2(n+m)+3$-tads)

\[
Lc \mathcal{F}(c, L) R^- R^+ + \mathcal{F}(c, L) R^- R^+ \subseteq
\]

(by 5.4)

\[
\{ Lc \mathcal{F}(c, L) \} R^- R^+ + \mathcal{F}(c, L) R^- R^+ \subseteq
\]

\[
\mathcal{F}(c, L) R^- R^+
\]

(since $\mathcal{F}(c, L)$ is an ideal of the subalgebra $L$ of $V^-(c)$ by 5.4, hence $\{ Lc \mathcal{F}(c, L) \} \subseteq \mathcal{F}(c, L)$).

Therefore $d = Q_L Q_0 Q_0 a b e \in abf e R^- R^+ \subseteq \mathcal{F}(c, L) R^- R^+$, and for some $p \geq 1$,

\[
d \in \sum_{k=1}^{p} \mathcal{F}(c, L)(V^- V^+)^k.
\]

Now, take a $g = f_{4p+3} \in \mathcal{H}_{4p+3}(V^+)$ as before. Then, by (3),

\[
h = Q_L Q_0 g = dbgbd = dbgbd^* \in \sum_{i=1}^{p} Q_L (Q_{V^-} Q_{V^+})^i Q_0 g
\]

\[
+ \sum_{k,l \leq p} \{ \mathcal{F}(c, L)(V^- V^+)^k bgb(V^+ V^-)^l \mathcal{F}(c, L) \}
\]

\[
\subseteq Q_L V^- + \{ \mathcal{F}(c, L) V^- V^- \mathcal{F}(c, L) \} \subseteq
\]

(since $g$ eats embedded $4p+3$-tads)

\[
L + \{ \mathcal{F}(c, L) V^- V^- \mathcal{F}(c, L) \} \subseteq L
\]

(by 5.4).

Thus $h \in L$, but $h \equiv d^{(11,b)} \mod K_0$ is again a $b$-modulus for $K$ by 6.1.

Therefore $L = V^+$ by 4.3, contradicting properness of $L$. ■
This result will allow us to determine the non homotope-PI maximal (weak) primitizers in terms of the associative envelopes. Note first that if \( V \) is a primitive Jordan pair (hence a strongly prime one 0.10), and \( V \) is not homotope-PI, then it is special (since exceptional, strongly prime systems are homotope-PI (Zelmanov, 1983b), and \( i \)-special, non homotope-PI, strongly prime systems are hermitian, hence special (D’Amour, 1992)). This happens in particular whenever the Jordan pair \( V \) has a primitizer which is not homotope-PI in \( V \).

**Proposition 6.3.** Let \( V \) be a special Jordan pair, \( K \subseteq V^\sigma \) be a maximal-(weak) primitizer for the modulus \( (a, b) \in V^\sigma \times V^{-\sigma} \), and take \( R=(R^+, R^-) \) an associative \( * \)-envelope of \( V \). If \( K \) is not homotope-PI in \( V \), then there exists a maximal right ideal \( M \subseteq R^+ \) of \( R \) which is \( (a, b) \)-modular and satisfies \( K = M \cap V^\sigma \).

**Proof.** Assume \( \sigma = + \). Since \( K \) is not homotope-PI in \( V \), it follows from 6.2 that \( N = K + KR^- R^+ \neq R^+ \). Set
\[
B = \{ r \in R^+ | (x - abx)br \in N \text{ for all } x \in V^+ \}.
\]
This is obviously a right ideal of \( R \). Moreover, if \( r \in R^+ \) and \( x \in V^+ \),
\[
(x - abx)b(r - abr) = (x - abx - xba + abxba)br = (B_{a,b,x})br \in KbR^+ \subseteq N,
\]
whence \( r - abr \in B \), and \( B \) is \( (a, b) \)-modular.

On the other hand, if \( k \in K \), \( x \in V^+ \),
\[
(x - abx)bk = \{ x, b, k \} - \{ a, Q_h x, k \} + kb(x - abx) \in K + Kr^- R^+ = N,
\]
hence \( K \subseteq B \cap V^+ \). Now it is straightforward that \( B \cap V^+ \) is an \( (a, b) \)-modular inner ideal, hence from the maximality of \( K \) it follows that either \( K = B \cap V^+ \) or \( B \cap V^+ = V^+ \).

In the latter case we have \( B = R^+ \), hence for all \( x \in V^+ \), \( (x - abx)br^+ \subseteq N \). In particular, taking \( x = a \), \( xba - Q_h x, x \in N \) for all \( x \in V^+ \). Then \( x - abx = B_{a,b,x} + (xba - Q_h x) \in K + N = N \) for all \( x \in V^+ \), hence \( r - abr \in N \) for all \( r \in R^+ \), and \( N \) is \( (a, b) \)-modular. Again \( K \subseteq N \cap V^+ \) implies that either \( K = N \cap V^+ \) or \( V^+ = V^+ \cap N \), but the latter is impossible since \( N \) is proper.

Thus, in any case we find an \( (a, b) \)-modular proper right ideal \( N \) of \( R \) with \( K = R^+ \cap N \). Now, applying Zorn’s lemma we can get a maximal
right ideal $M$ of $R$ which is $(a, b)$-modular and contains $K$. By properness of $M$, this implies $K = M \cap V^+$, and we are done.

**Remark 6.4.** Similar results are valid for Jordan triple systems, making the obvious changes in the proofs of 6.2 and 6.3. Thus we get:

Let $J$ be a special Jordan triple system, $K \subseteq J$ a maximal-(weak) primitizer for the modulus $(a, b) \in J \times J$, and let $R$ be an associative $*$-envelope of $J$. If $K$ is not homotope-PI in $J$, there exists a maximal right ideal $M$ of $R$ which is $(a, b)$-modular, and satisfies $K = M \cap J$.

We can now prove Anquela and Cortés’ conjecture for weak primitizers of a Jordan pair $V$ which are not homotope-PI in $V$.

**Corollary 6.5.** In the conditions of 6.3, if $K$ is weakly $(a, b)$-modular, it is $(a, b)$-modular.

**Proof.** This follows since if $M$ is $(a, b)$-modular in $R$, $V^+ \cap M$ is $(a, b)$-modular in $V$.

To prove the maximality among all inner ideals of maximal-modular inner ideals which are not homotope-PI in a Jordan system we will apply the following result:

**Proposition 6.6.** Let $J$ be a special Jordan system (pair or triple) with associative $*$-envelope $R$, let $K$ be primitizer for $J$ with modulus $(a, b)$, and let $M$ be a maximal $(a, b)$-modular right ideal of $R$ with $J \cap M = K$. If $K$ is not homotope-PI in $J$, then any right ideal $N$ of $R$ with $K \subseteq N \cap J$ has: $N = R$ or $N \subseteq M$.

**Proof.** We do the proof for Jordan pairs $J = V = (V^+, V^-)$, and the same argument with the obvious modifications apply to Jordan triple systems.

By taking the opposite pair if necessary, we can assume $K \subseteq V^+$, and $N, M \subseteq R^+$. Now, if $N \subseteq M$, by the maximality of $M$ we have $R^+ = M + N$, hence we can find $x \in M$, $y \in N$ with $a = x + y$.

On the other hand, since $K$ is nonzero (because it is not homotope-PI in $V$) the ideal $I$ generated by $K$ in $R^+$ is nonzero. Since $K$ is a primitizer, and $0 \neq K \subseteq I^+ \cap V^+$, we get $I^+ \cap V^+ = V^+$, and hence $a \in I^+ \cap V^+$. In particular, with the notations of 6.2, there is $n \geq 0$ such that

$$a \in \sum_{k,l \leq n} (V^+ V^-)^k K(V^- V^+)^l.$$  \hspace{1cm} (1)
Moreover, we can choose \( n \) such that
\[
x \in \sum_{i \leq n} V^+ (V^- V^+)^i.
\]

Also, as in the proof of 6.2, we can find a hearty \( 4n + 5 \)-tad eater \( f \in V^+ \) with \( f \equiv a \mod K \).

Then we have, \( abfba = xbfbc + ybfbc \), and, if \( k \leq n \), \( \{ xbfb (V^+ V^-)^k K \} \subseteq M \cap \bigcup_{i=0}^{2n} V^+(V^- V^+)^i \{ V^+ V^-)^k K \} \) (by (2) and \( K \subseteq M \) \( \subseteq M \cap V^+ \) (since \( f \) eats \( 4n + 5 \)-tads) \( = K \). Then
\[
xbfbc \subseteq \sum_{k, l \leq n} xbfb (V^+ V^-)^k K (V^- V^+)^l \\
\subseteq \sum_{i, k, l \leq n} \{ xbfb (V^+ V^-)^k K \} (V^- V^+)^l + KR^+ R^-
\]
\[
\subseteq KR^+ R^- \subseteq N \quad (\text{since } K \subseteq N)
\]

Therefore, \( d = Q_a Q_b f = abfba = xaba + yaba \in N \cap V^+ \). But \( d \equiv d^{(3, b)} \mod K \) by 6.1 (note that \( K = K_0 \) by modularity), hence \( d \) is a \( b \)-modulus for \( K \) by 0.8(2), and therefore \( N \cap V^+ = V^+ \) by 4.3, hence \( N = R^+ \).

We can now prove Hogben and McCrimmon’s conjecture for maximal primitizers of Jordan systems which are not homotope-PI in the system.

**Proposition 6.7.** Let \( J \) be a Jordan pair or triple system. Then every maximal primitizer of \( J \) which is not homotope-PI in \( J \) is maximal among all inner ideals of \( J \).

**Proof.** We do it for Jordan pairs \( J = V = (V^+, V^-) \), and the same proof works for triple systems.

Let \( K \subseteq V^n \) be a maximal primitizer, for the modulus \( (a, b) \in V^n \times V^{-n} \), which is not homotope-PI in \( V \). As usual, we can assume \( \sigma = + \). Suppose that \( K \subseteq L \) for a proper inner ideal \( L \subseteq V^+ \).

Now, since \( K \) is not homotope-PI in \( V \), \( V \) itself is not homotope-PI, and thus it is special. Take an associative \(*\)-envelope \( R = (R^+, R^-) \) of \( V \). Then \( K = M \cap V^+ \) for a maximal \( (a, b) \)-modular right ideal \( M \subseteq R^+ \) by 6.3. On the other hand, \( N = L + LR^- R^+ \) is a right ideal of \( R \) with \( K \subseteq L \subseteq N \cap V^+ \), and it is proper by 6.2. Thus \( N \subseteq M \) by 6.6, hence \( K \subseteq L \subseteq N \cap V^+ \subseteq M \cap V^+ = K \), and \( K = L \).
7. MAIN THEOREMS

We have now all the ingredients for the proofs of Anquela and Cortés’ conjecture and Hogben and McCrimmon’s conjecture.

**Theorem 7.1.** Maximal weakly modular inner ideals of Jordan pairs and triple systems are modular.

*Proof.* Consider first a Jordan pair \( V \), and suppose that \( K \) is a maximal-weakly modular inner ideal for some modulus \((a, b)\). Take \( I = \text{Core}(K) \). Then \( K = K/I^+ \) is clearly a maximal weakly \((a + I^+, b + I^-)\)-modular inner ideal in \( \overline{V} = V/I \) with zero core, and moreover, \( K \) is \((a, b)\)-modular if and only if \( K \) is \((a + I^+, b + I^-)\)-modular. Therefore we can assume \( \text{Core}(K) = 0 \), and thus \( K \) is a maximal weak primitizer.

If \( K \) is homotope-PI in \( V \), then \( V \) has finite capacity by 3.5, hence \( K \) is \((a, b)\)-modular by 2.9. If \( K \) is not homotope-PI in \( V \), then it is \((a, b)\)-modular by 6.5. Thus, in any case, \( K \) is \((a, b)\)-modular.

Now consider a Jordan triple system \( T \), and a maximal-weakly modular inner ideal \( K \) for some modulus \((a, b)\). Then \( K \subseteq V(T)^+ = T \) is maximal-weakly modular for the modulus \((a, b)\) by 0.12(b). Thus it is \((a, b)\)-modular by the pair case, hence \( K \) is \((a, b)\)-modular in \( T \) by 0.12(a).

**Theorem 7.2.** Maximal-modular inner ideals of Jordan pairs and triple systems are maximal among all inner ideals.

*Proof.* As in the proof of 7.1 we consider first the case of a Jordan pair \( V \), and we can assume that the maximal-modular inner ideal \( K \) has \( \text{Core}(K) = 0 \). Thus, if \( K \) is homotope-PI in \( V \), then \( V \) has finite capacity by 3.5, hence \( K \) is maximal among all inner ideals by 2.10. If \( K \) is not homotope-PI in \( V \), then it is maximal among all inner ideals by 6.7. Thus, in any case, \( K \) is maximal among all inner ideals.

Now consider a Jordan triple system \( T \), and a maximal-modular inner ideal \( K \) for some modulus \((a, b)\). Then \( K \subseteq V(T)^+ = T \) is maximal-modular for the modulus \((a, b)\) by 0.12(b). Thus it is maximal among all inner ideals of \( V(T) \) by the previous case, hence \( K \) is maximal in \( T \).

**Theorem 7.3.** Let \( J \) be a primitive Jordan pair or triple system and let \( K \) be a maximal primitizer of \( J \), then one of the following holds:
(a) If $K$ is homotope-PI in $J$, then $J$ is simple of finite capacity, and $K$ is a maximal inner ideal.

(b) If $K$ is not homotope-PI in $J$, then $J$ is special, and if $(R, \ast)$ is a $\ast$-tight associative envelope, there exists a unique maximal primitizer $M$ of $R$ with modulus $(a, b) \in J \times J$ such that $K = J \cap M$.

Proof. First consider the case where $K$ is not homotope-PI in $J$, then $J$ is not homotope-PI, and therefore, as mentioned before 6.3, it is special. The existence of $M$ is 6.3 (and 6.4, for triple systems), the fact that this $M$ is a maximal primitizer follows from tightness, and the uniqueness easily follows from 6.6.

Next, suppose that $K$ is homotope-PI in $J$. If $J$ is a Jordan pair, then the result follows from 3.5 and 2.10, so we only need to examine the triple case. Consider then the double $V(J)$, and with the notations of 0.12, the inner ideal $K^+ \subseteq V(J)^+$. As mentioned in 0.12(c), $K^+$ is not necessarily a primitizer of $V(J)$ (although it is maximal $(a^+, b^\ast)$-modular), but setting $I = \text{Core}(K^+)$, $K^+ = K^+/I^+$ is a maximal primitizer of the tight double $W = V(J)/I$, and it is still homotopr-PI in $W$. Thus the pair case implies that $W$ is simple of finite capacity, and the result follows if $I = 0$, so we can assume $I \neq 0$. Now consider the polarized triple $T(W) = W^+ \oplus W^-$, and the monomorphism $\tau : J \rightarrow T(W)$ given by $\tau(x) = \pi_+(x) \oplus \pi_-(x)$, where $\pi_0 : J \rightarrow J/I^0$ is the projection. Now set $I = I^+ + I^- \subseteq J$. This is a nonzero ideal of $J$, hence $V(I) = (I, \bar{I})$ is a nonzero ideal of $V(J)$, and $V(I)/I$ is a nonzero ideal of the simple pair $W$. Thus $W^0 = I/I^0 = \pi_0(I^0)$, hence $\tau(I) = \pi_+(I^-) \oplus \pi_-(I^-) = W$. Therefore $\tau$ is surjective, and $J$ is isomorphic to the triple $T(W)$, which is simple of finite capacity since $W$ is (see D’Amour and McCrimmon, 2000, p. 34).

Remark 7.4. The previous result suggest the question of whether all inner ideals described in (a) and (b) are indeed maximal modular. More concretely, we leave open the following two questions

(c) Is every (maximal) inner ideal in a simple Jordan system of finite capacity modular?

and

(d) If $J$ is a special Jordan system with associative $\ast$-envelope $R$, and $M$ is a maximal primitizer of $R$ with modulus $(a, b) \in J \times J$, is $M \cap J$ a maximal primitizer of $J$ for the modulus $(a, b)$?
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