Computational approach to the simplicity of $f_4(O_s, -)$ in the characteristic two case

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Abstract

We present in this paper a computational approach to the study of the simplicity of the derivation Lie algebra of the quadratic Jordan algebra $H_3(O_s, -)$, denoted by $f_4(O_s, -)$, when the characteristic of the base field is two. We will show not only a collection of routines designed to find identities and construct principal ideals but also a philosophy of how to proceed studying the simplicity of a Lie algebra. We have first implemented the quadratic Jordan structure of $H_3(O_s, -)$ into the computer system Mathematica (Computing the derivation Lie algebra of the quadratic Jordon Algebra $H_3(O_s, -)$ at any characteristic, preprint, 2001) and then determined the generic expression of an element of the Lie algebra $f_4(O_s, -) = \text{Der}(H_3(O_s, -))$ (see (41)). Once the structure of $f_4(O_s, -)$ is completely described, it is time to analyze the simplicity by using the strategy mentioned. If the characteristic of the base field is not two, the Lie algebra is simple, but if the characteristic is two, the Lie algebra is not simple and there exists only one proper nonzero ideal $I$ which is 26 dimensional and simple as a Lie algebra. In order to prove this last affirmation, we have used again the set of routines to show the simplicity of the ideal and that it is isomorphic to $f_4/I$, which is also a simple Lie algebra. This isomorphism is constructed from a computed Cartan decomposition of both Lie algebras.

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1. Previous results

A quadratic Jordan algebra with 1 over an arbitrary commutative ring $\Phi$ with 1 is a triple $(J, U, 1)$, where $J$ is a unital left $\Phi$-module, 1 a distinguished element of $J$, and $U$ is a mapping $a \mapsto U_a$ of $J$ into $\text{Hom}_\Phi(J, J)$ satisfying the following axioms: (1) $U$ is $\Phi$-quadratic, that is, $U_{xa} = x^2 U_a$, $x \in \Phi$, $a \in J$ and $U_{a,b} := U_{a+b} - U_a - U_b$ is $\Phi$-bilinear in $a$ and $b$; (2) $U_1 = 1$; (3) $U_a U_b U_y = U_{U_y(b)}$; (4) $U_x(T(y, x, z)) = T(x, y, U_x(z))$, $\forall x, y, z \in J$ where $T(y, x, z) := U_{y,z}(x)$ and (5) conditions (3) and (4) hold in every scalar extension of $J$.

If $(O_s, -)$ is a split Cayley algebra (an split octonions algebra) over $F$ with $-$ as its canonical involution, then $H_3(O_s, -)$ is the set of matrices

$$X = \begin{pmatrix} \lambda_1 & a & b \\ \bar{a} & \lambda_2 & c \\ \bar{b} & \bar{c} & \lambda_3 \end{pmatrix} \quad \text{with} \quad \lambda_i \in F, \ a, b, c \in O_s.$$  \hspace{1cm} (1)

It is usual to express the generic element (1) of $H_3(O_s, -)$ as

$$X = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 + X_1(a) + X_2(b) + X_3(c),$$  \hspace{1cm} (2)

where the definitions of $E_i$ and $X_i$ are the obvious ones. A basis of $H_3(O_s, -)$ is the following:

$$\mathcal{B} := \{E_1, E_2, E_3, X_1(e_i)_{i=1,\ldots,8}, X_2(e_i)_{i=1,\ldots,8}, X_3(e_i)_{i=1,\ldots,8}\},$$  \hspace{1cm} (3)

where $e_i$, with $i = 1, \ldots, 8$, is the standard basis of the alternative algebra $O_s$, seen as the Zorn’s matrices algebra. In order to define a quadratic Jordan structure in $H_3(O_s, -)$ we use the McCrimmon’s equations [6]:

$$U_{a[ii]}b[ii] = aba[ii],$$  \hspace{1cm} (4)

$$U_{a[ij]}b[ii] = \bar{a}ba[jj],$$  \hspace{1cm} (5)

$$U_{a[ij]}b[ij] = a\bar{b}a[ij],$$  \hspace{1cm} (6)

$$T(a[ii], b[ij], c[ij]) = abc[ij],$$  \hspace{1cm} (7)

$$T(a[ii], b[ij], c[ji]) = (abc + \overline{abc})[ii],$$  \hspace{1cm} (8)

$$T(a[ii], b[ij], c[jk]) = abc[ik],$$  \hspace{1cm} (9)

$$T(a[ii], b[ii], c[ij]) = abc[ij],$$  \hspace{1cm} (10)

$$T(a[ij], b[jj], c[jk]) = abc[ik],$$  \hspace{1cm} (11)

$$T(a[ij], b[ji], c[ik]) = a(bc)[ik],$$  \hspace{1cm} (12)

$$T(a[ij], b[jk], c[ij]) = (a(bc) + \overline{a(bc)})[ii],$$  \hspace{1cm} (13)
where $a_{ij}$ represents an element of $H_3(C, -)$ filled with zeros except for an octonion $a$ at the $(i, j)$ position, $T(a, b, c) = U_{a, c}(b)$ and it is understood that all the $U$ formulas not covered by these and $a_{ji} = \bar{a}_{ij}$ are 0. A derivation in this context (see [8, 1.4, p. 3]) is a linear map $D$ satisfying:

$$D(1) = 0, \quad D(U_a(y)) = T(D(x), y, x) + U_x(D(y)) \quad \forall x, y.$$  \hfill (14)

It is easy to see that the vector space of all derivations of $H_3(O_s, -)$ is a Lie algebra with respect to the Lie product $[D, D'] = DD' - D'D$. Then we write $f_4(O_s, -)$ to be the Lie algebra of the derivations of the quadratic Jordan algebra $H_3(O_s, -)$. It is important to note that this construction does not depend on the characteristic of the base field. We have implemented the structure of $H_3(O_s, -)$ into a computer system and, by using the multiplications rules with $U$ and $T$ and the conditions that define a derivation, we have determined the generic matrix of a derivation in $f_4(O_s, -)$ in the basis $B$ (see [3]). We have obtained (see [3]) that the generic element of $f_4(O_s, -)$, without any restriction on the characteristic, is as (41) with dim $f_4(O_s, -) = 52$.

2. Some results and routines for the simplicity

If $L$ is a finite-dimensional Lie algebra, we are going to use the following notation, where $B$ is, perhaps, a basis of $L$ and $K$ is the base field:

$$\forall x \in L - \{0\}, \exists \lambda \neq 0 \in K, x_1, x_2, \ldots, x_n \in B$$

such that $[[\ldots[x, x_i], x_2], \ldots, x_n] = \lambda x_i$

$$\equiv \forall x \in L - \{0\}, \exists i: x \vdash x_i \in B.$$  \hfill (15)

It will be a crucial point the fact that in a Lie algebra we have that $\forall x \in L - \{0\}, \exists i: x \vdash x_i \in B$. To prove this we have to study different possibilities for the parameters of a generic element of the Lie algebra. This could be represented by several trees of identities. If these trees do exist and the ideal generated by each element of the basis is the whole Lie algebra, then the algebra is simple:

**Lemma 1.** Let $L$ be a Lie algebra and $B = \{x_i\}_{i=1}^n$ a basis. If

1. $\forall x \in L - \{0\}, \exists i \in \{1, \ldots, n\}: x \vdash x_i \in B$ (trees of identities),
2. $\forall i \in \{1, \ldots, n\}: x_i \vdash x, \forall x \in L$ (Principal ideals are the whole algebra),

then $L$ is simple.

If the principal ideals are not the whole algebra, but they are the same ideal, we can conclude that the mentioned ideal is minimal. The routine used to compute the principal ideals is based on the chain

$$\{x\} \hookrightarrow \{[x, x_i]\}_{i=1}^{S_2} \hookrightarrow \{[[x, x_i], x_j]\}_{i, j=1}^{S_2} \hookrightarrow \{[[[x, x_i], x_j], x_k]\}_{i, j, k=1}^{S_2} \hookrightarrow \ldots \subset f_4$$

In each step the Lie brackets by the basis elements are included. Then, the linear independence of the set obtained is analyzed. The routine stops when the set is the whole algebra or when it does
not increase the dimension. The routine could be as

\[
\text{IdealGen[lis, car]} := \text{Module}\{k, \text{ideal}\}, \\
\text{ideal} = \text{Map[Comb, NZero]} \\
\text{RowReduce[Map[Coord, Map[Reco, lis]}, \\
\text{Modulus} \rightarrow \text{car}]]}; \\
\text{Print["Depth = ", 0 \rightarrow ", ideal]; dim[0] = 0; \\
\text{dim[1] = 1; } k = 1; \\
\text{While[} \text{dim}[k] < 52 \&\& \text{dim}[k] \neq \text{dim}[k - 1], \text{setd} = \text{ideal}; \\
k = k + 1; \text{Do[AppendTo[ideal,} \\
\text{Reco[ReleaseHold[setd[[j]]], x[i]]], \{i, 14\}, \{j, \text{Length[setd]}\}]; \\
\text{ideal} = \text{Map[Comb, NZero[RowReduce[} \\
\text{Map[Coord, ideal], Modulus \rightarrow car]]}; \\
\text{Print["Depth = ", k - 1 \rightarrow ", ideal] \\
\text{dim}[k] = \text{Length[ideal]}]; \text{Print["Ideal < ", } \\
\text{Map[Reco, lis], " > =", ideal, "with", "dimension = ", dim[k]]}
\]

where we denote by \(\{x_i\}\) a basis of the Lie algebra which, in our case, is 52 dimensional. It is assumed that we have defined previously \text{Reco}, to recognize a matrix as an element of the Lie algebra, \text{NZero}, to eliminate zeros from a list, \text{Coord} to obtain the coordinates of an element in the basis \(\{x_i\}\), and \text{Comb} to obtain a linear combination from a list of coordinates. In order to use the previous lemma, it is important to construct routines which look for identities. A pair of examples are

\[
\text{Do[} a = c[Z, x]; \text{If[Length[Variables[} a\text{]]] = 1 \&\& \\
\text{Length[Variables[} \text{Map[} \text{carac}_2, a, 2\text{]}\text{]]! = 0, Print[i], \{i, 52\}] \\
\text{Do[} a = c[Z, x], x_j]; \text{If[Length[Variables[} a\text{]]] = 1 \&\& \\
\text{Length[Variables[} \text{Map[} \text{carac}_2, a, 2\text{]}\text{]]! = 0, Print[i, ", ", j], \\
\{i, 52\}, \{j, 52\}]}
\]

where we have included the intention that the identity found is also true if the characteristic is two.

3. Studying the simplicity

We have applied this strategy to \(f_4(O_3, -)\) when the characteristic of the base field is two. The result has been that the Lie algebra is not simple and that there is, using Lemma 1, only one proper nonzero minimal ideal \(I\) which is 26 dimensional. We have used again Lemma 1 working in the ideal \(I\) seen as a Lie algebra. The result is that it is simple as a Lie algebra. The quotient Lie algebra \(f_4/I\) can be used also to prove the utility of the strategy of finding identities and computing the principal ideals by using the previous routines. As a result, the quotient Lie algebra is simple, that is, \(I\) is the unique proper nonzero ideal of \(f_4(O_3, -)\) if the characteristic of the base field is two.

We can, by using matrix (41), define a basis of \(f_4(O_3, -)\) in the characteristic two case,

\[
\mathcal{B} := \{x_i : i = 1, \ldots, 52\}.
\]
We have made this by sorting the parameters in (41) and making them to be one and zero the rest in each case.

Let us now study the ideal generated by, for example, $x_1$. If we make the products $[x_1, x_i]$, $i = 1, \ldots, 52$, with all the elements of the basis $B$ we obtain the linear independent set

$$\{x_1, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{24}, x_{25}\},$$  \hspace{1cm} (17)

where we have eliminated the zero elements and included $x_1$. By making products again of the elements in the previous set with all the basic elements we obtain the bigger linear independent set

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18},$$

$$x_{19}, x_{20}, x_{21}, x_{22}, x_{24}, x_{25}, x_{26} + x_{34} + x_{37}\}. \hspace{1cm} (18)$$

So the elements of the previous set are all the elements $[[x_1, x_i], x_j]$ or $[x, x]$ eliminating zeros and including $x_1$. If we increase the depth of the Lie products, we obtain the same set, that is, the ideal $I$ generated by $x_1$ is not the whole Lie algebra: the Lie algebra $f_4(O_s, -)$ is not simple if the characteristic of the base field is two. If we proceed this way, we obtain that the ideal generated by any of the first 25 elements of the basis $B$ is also $I$, and the remaining ones generate the whole algebra. We are going to prove that

$$I = \langle \{x_i\}_{i=1, \ldots, 25} \cup \{x_{26} + x_{34} + x_{37}\} \rangle$$  \hspace{1cm} (19)

is the only proper nonzero ideal of $f_4(O_s, -)$ in the characteristic two case. The following theorem proves that the ideal $I$ is minimal: the ideal generated by a generic element is the whole algebra or it contains $I$:

**Theorem 1.** Let $f_4(O_s, -)$ be the Lie algebra of the derivation of the quadratic Jordan algebra $H_3(O_s, -)$ without any restriction on the characteristic. Then for all $0 \neq X \in f_4(O_s, -)$ there exists $i \in \{1, \ldots, 52\}$ such that $x_i \in (X)$, where $(X)$ denotes the ideal generated by $X$.

**Proof.** If $X$ is a generic element of $f_4(O_s, -)$, as in (41), we have the identity $[[[X, x_2], x_{11}], x_{45}] = \delta_2 x_{19}$. If $\delta_2 \neq 0$, the basic element $x_{19}$, and the ideal generated by $x_{19}$, that is, $I$, is contained in the ideal generated by $X$. If $\delta_2 = 0$, we use the identity $[[X, x_{38}], x_{45}] = -\epsilon_1 x_{45}$ to have that $x_{45} \in (X)$ if $\epsilon_1 \neq 0$. If $\delta_2$ and $\epsilon_1$ are zero then $[[X, x_{28}], x_{38}] = -\rho_5 x_{38}$. This equation allows us to conclude that $x_{38} \in (X)$ if $\rho_5 \neq 0$. All the rest identities are

- If also $\rho_5 = 0$ then $[[X, x_{27}], x_{28}] = -\rho_4 x_{28}$; if also $\rho_4 = 0$ then $[[X, x_{27}], x_{29}] = -\rho_6 x_{27}$;
- if also $\rho_6 = 0$ then $[[X, x_{27}], x_{31}] = -\rho_2 x_{27}$; if also $\rho_2 = 0$ then $[[X, x_{27}], x_{32}] = -\rho_3 x_{27}$;
- if also $\rho_3 = 0$ then $[[X, x_{27}], x_{30}] = \alpha_1 x_3$; if also $\alpha_1 = 0$ then $[[X, x_3], x_{30}] = -\rho_1 x_3$;
- if also $\rho_1 = 0$ then $[[X, x_{27}], x_{33}] = -\eta_2 x_{27}$; if also $\eta_2 = 0$ then $[[X, x_{27}], x_{48}] = -\eta_3 x_{27}$;

\[1\] The characteristic of the base field is two.
if also $\eta_3 = 0$ then $[[X, x_{27}], x_{36}] = -\gamma_6 x_{20}$; if also $\gamma_6 = 0$ then $[[X, x_{20}], x_{36}] = \chi_1 x_{20}$;
if also $\lambda_1 = 0$ then $[[X, x_{27}], x_{51}] = \eta_4 x_{27}$; if also $\eta_4 = 0$ then $[[X, x_{27}], x_{49}] = \beta_5 x_{17}$;
if also $\beta_5 = 0$ then $[[X, x_{17}], x_{49}] = -\varepsilon_3 x_{17}$; if also $\varepsilon_3 = 0$ then $[[X, x_{26}], x_{27}] = \eta_3 x_{32}$;
if also $\eta_5 = 0$ then $[[X, x_{27}], x_{34}] = \beta_4 x_{17}$; if also $\beta_4 = 0$ then $[[X, x_{1}], x_{27}] = \gamma_2 x_{20}$;
if also $\gamma_2 = 0$ then $[[X, x_{10}], x_{27}] = -\alpha_6 x_{18}$; if also $\alpha_6 = 0$ then $[[X, x_{4}], x_{42}] = -\delta_5 x_4$;
if also $\delta_5 = 0$ then $[[X, x_{16}], x_{35}] = -\phi_2 x_{16}$; if also $\phi_2 = 0$ then $[[X, x_{18}], x_{50}] = \varepsilon_4 x_{18}$;
if also $\varepsilon_4 = 0$ then $[[X, x_{26}], x_{28}] = -\gamma_7 x_{18}$; if also $\gamma_7 = 0$ then $[[X, x_{1}], x_{28}] = \beta_3 x_{17}$;
if also $\beta_3 = 0$ then $[[X, x_{10}], x_{28}] = -\alpha_7 x_{18}$; if also $\alpha_7 = 0$ then $[[X, x_{3}], x_{43}] = -\delta_6 x_5$;
if also $\delta_6 = 0$ then $[[X, x_{6}], x_{51}] = \beta_2 x_{18}$; if also $\beta_2 = 0$ then $[[X, x_{1}], x_{51}] = -\gamma_5 x_{18}$;
if also $\gamma_5 = 0$ then $[[X, x_{6}], x_{18}] = -\alpha_8 x_{18}$; if also $\alpha_8 = 0$ then $[[X, x_{6}], x_{39}] = -\phi_1 x_6$;
if also $\phi_1 = 0$ then $[[X, x_{1}], x_{29}] = \gamma_8 x_{18}$; if also $\gamma_8 = 0$ then $[[X, x_{1}], x_{31}] = \beta_1 x_{16}$;
if also $\beta_1 = 0$ then $[[X, x_{1}], x_{10}] = -\delta_1 x_{18}$; if also $\delta_1 = 0$ then $[[X, x_{27}], x_{27}] = -\eta_1 x_{27}$;
if also $\eta_1 = 0$ then $[[X, x_{28}], x_{28}] = -\delta_2 x_{28}$; if also $\delta_2 = 0$ then $[[X, x_{4}], x_{6}] = \gamma_4 x_{18}$;
if also $\gamma_4 = 0$ then $[[X, x_{1}], x_{33}] = -\alpha_6 x_{16}$; if also $\alpha_6 = 0$ then $[[X, x_{2}], x_{42}] = \delta_3 x_4$;
if also $\delta_3 = 0$ then $[[X, x_{1}], x_{35}] = \beta_6 x_{16}$; if also $\beta_6 = 0$ then $[[X, x_{1}], x_{6}] = \gamma_3 x_{16}$;
if also $\gamma_3 = 0$ then $[[X, x_{18}], x_{18}] = \zeta x_{18}$; if also $\zeta = 0$ then $[[X, x_{16}], x_{16}] = \zeta x_{18}$;
if also $\zeta = 0$ then $[[X, x_{31}], x_{27}] = \phi_4 x_{27}$; if also $\phi_4 = 0$ then $[[X, x_{3}], x_{3}] = \delta_3 x_{27}$;
if also $\delta_3 = 0$ then $[[X, x_{39}], x_{6}] = \alpha_3 x_6$; if also $\alpha_3 = 0$ then $[[X, x_{30}], x_{29}] = -\psi_2 x_{29}$;
if also $\psi_2 = 0$ then $[[X, x_{3}], x_{3}] = \alpha_3 x_3$; if also $\alpha_3 = 0$ then $[[X, x_{26}], x_{3}] = -\alpha_2 x_3$;
if also $\alpha_2 = 0$ then $[[X, x_{3}], x_{3}] = -\beta_1 x_{18}$; if also $\beta_1 = 0$ then $[[X, x_{7}], x_{15}] = -\gamma_1 x_{15}$;
if also $\gamma_1 = 0$ then $X = \delta_4 x_{29}$.

Let us consider the ideal $I$ as a Lie algebra. Is it simple? We can take the basis $\mathfrak{B}_I := \{y_i : i = 1, \ldots, 26\}$ of $I$, where

$$y_i := x_i, \quad i = 1, \ldots, 25 \quad \text{and} \quad y_{26} := x_{26} + x_{34} + x_{37}. \tag{20}$$

If we take $y_1$ and make Lie products with the rest of the elements of $\mathfrak{B}_I$ we obtain, at a first step, the set

$$\{y_1, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23}, y_{24}, y_{25}\}, \tag{21}$$

where we have eliminated the null elements and added $y_1$. If we make Lie brackets again we obtain all the elements of the basis. That is, the ideal generated by $y_1$ is $I$. In the same way, the ideal generated by $y_i$, $i = 2, \ldots, 26$, is also $I$. Let $Y$ be any nonzero element $Y = \sum_{i=1}^{26} \alpha_i y_i$. We only have then to repeat a similar argument as we did to prove that $I$ is minimal to conclude that $I$ is simple as a Lie algebra because in each step we have that the ideal generated by some $y_1$, that is, $I$, is contained in the one generated by $Y$. As $[[[Y, y_1], y_2], y_{10}] = \alpha_{11} y_2$, if $\alpha_{11} \neq 0$, we are done. If $\alpha_{11} = 0$, then proceed as follows:

If $\alpha_{11} = 0$ then $[[Y, y_2], y_{10}] = \alpha_{13} y_{10}$; if also $\alpha_{3} = 0$, $[[Y, y_2], y_{12}] = \alpha_{15} y_2$;
if also $\alpha_{15} = 0$, $[[Y, y_2], y_{13}] = \alpha_{16} y_2$; if also $\alpha_{16} = 0$, $[[Y, y_2], y_{14}] = \alpha_{17} y_2$;
if also $\alpha_{17} = 0$, $[[Y, y_2], y_{19}] = \alpha_{18} y_2$; if also $\alpha_{18} = 0$, $[[Y, y_1], y_{19}] = \alpha_{10} y_2$;
if also $\alpha_{10} = 0$, $[[Y, y_2], y_7] = \alpha_{20} y_{19}$; if also $\alpha_{20} = 0$, $[[Y, y_1], y_{23}] = \alpha_{12} y_2$;
Theorem 2. \( I \) is a 26-dimensional simple Lie algebra with a two-dimensional Cartan subalgebra and Cartan decomposition as follows:

\[
 I = H \oplus I_x \oplus I_\beta \oplus I_{x+\beta}, \quad H = \langle y_1, y_{26} \rangle,
\]

\[
 I_x = \langle y_{18}, \ldots, y_{25} \rangle, \quad I_\beta = \langle y_2, \ldots, y_9 \rangle, \quad I_{x+\beta} = \langle y_{10}, \ldots, y_{17} \rangle,
\]

\[
 \alpha : H \rightarrow F; \quad \alpha(y_1) = 1, \quad \alpha(y_{26}) = 0,
\]

\[
 \beta : H \rightarrow F; \quad \beta(y_1) = 0, \quad \alpha(y_{26}) = 1,
\]

\[
 \alpha + \beta : H \rightarrow F; \quad (\alpha + \beta)(y_1) = 1, \quad (\alpha + \beta)(y_{26}) = 1,
\]

\[
 \operatorname{ad}(y_1)|_{I_\beta} = 1_{I_\beta}, \quad \operatorname{ad}(y_1)|_{I_{x+\beta}} = 0, \quad \operatorname{ad}(y_1)|_{I_{x+\beta}} = 1_{I_{x+\beta}},
\]

\[
 \operatorname{ad}(y_{26})|_{I_x} = 0, \quad \operatorname{ad}(y_{26})|_{I_\beta} = 1_{I_\beta}, \quad \operatorname{ad}(y_{26})|_{I_{x+\beta}} = 1_{I_{x+\beta}}.
\]

We are going to prove that the quotient \( f_4(O_s, -)/I \) is also simple, that is, the ideal \( I \) is not only minimal, but also maximal: it is unique among the proper nonzero ideals. We can extend the basis \( \mathcal{B}_I \) with \( y_i := x_i, \) \( i = 27, \ldots, 52, \) to construct a basis of \( f_4(O_s, -). \) Thus, a basis of \( f_4(O_s, -)/I \) is

\[
 \mathcal{B}_c := \{ \overline{y_i} = y_i + I : i = 27, \ldots, 52 \}.
\]

By taking the projection \( P \) onto the subspace generated by \( \{ y_{27}, \ldots, y_{52} \}, \) it is easy to define a Lie bracket on \( f_4(O_s, -)/I \) as

\[
 \langle x + I, y + I \rangle := P([x, y]) + I, x, y \in f_4(O_s, -).
\]

Then, let us take \( \overline{y_{27}}. \) Performing the products with the rest of the elements of \( \mathcal{B}_c \) we obtain, \( \overline{y_{27}} \) included, the linear independent set

\[
 \{ \overline{y_{27}}, \overline{y_{28}}, \overline{y_{29}}, \overline{y_{31}}, \overline{y_{32}}, \overline{y_{33}}, \overline{y_{34}}, \overline{y_{48}}, \overline{y_{51}}, \overline{y_{52}} \}.
\]
If we add to this set the one obtained by making again products with the elements in \( B \) except \( y_{45} \), which is included in the next level of depth. Thus, the ideal generated by \( y_{27} \) in the quotient \( f_4(O_5, -) / I \) is the whole algebra. The same occurs with the rest of the elements of the basis. We repeat the previous argument to prove that any element of the algebra generates the quotient. If \( Z \) is a generic element in the quotient \( Z = \sum_{i=27}^{52} y_i \), we have that \( [Z, y_{27}] = y_{45} \). If \( y_{45} \neq 0 \), we are done, but, if \( y_{45} = 0 \), then we can proceed as we did with \( I \), by finding a similar set of identities. We can affirm then that the quotient is simple and, as a consequence, \( I \) is maximal and it is the only proper nonzero ideal in \( f_4(O_2, -) \), if the characteristic is two.\(^2\)

**Theorem 3.** Let \( f_4(O_5, -) \) be the Lie algebra of derivations of the quadratic Jordan algebra \( H_3(O_5, -) \) in the split characteristic two case. Then,

1. A generic element of \( f_4(O_5, -) \) is as (41), in the basis (3).
2. The Lie algebra \( f_4(O_5, -) \) is not simple:
   - (a) There exists only one proper nonzero ideal, \( I \), in \( f_4(O_5, -) \), see (18), which is 26 dimensional (its structure is given in Theorem 2).
   - (b) \( I \) and the quotient algebra \( W := f_4(O_5, -) / I \) are isomorphic simple Lie algebras.

**Proof.** After all, we only have to prove that \( I \) and \( W \) are isomorphic. For that, we are going to show first that they have a similar Cartan decomposition and secondly we will use this decomposition to define an isomorphism. Using the previous definitions it is easy to check that the quotient has a two-dimensional Cartan decomposition:

\[
W = K \oplus W_x \oplus W_\beta \oplus W_{x+\beta}, \quad K = \langle y_{34}, y_{37} \rangle,
\]

\[
W_x = \langle y_{28}, y_{46}, y_{48}, y_{43}, y_{41}, y_{34}, y_{31}, y_{52} \rangle,
\]

\[
W_\beta = \langle y_{35}, y_{49}, y_{27}, y_{50}, y_{30}, y_{45}, y_{36}, y_{42} \rangle,
\]

\[
W_{x+\beta} = \langle y_{29}, y_{47}, y_{33}, y_{44}, y_{40}, y_{38}, y_{32}, y_{51} \rangle,
\]

\[
x : K \to F; \quad x(y_{34}) = 1, \quad x(y_{37}) = 0,
\]

\[
\beta : K \to F; \quad \beta(y_{34}) = 0, \quad \beta(y_{37}) = 1,
\]

\[
x + \beta : K \to F; \quad (x + \beta)(y_{34}) = 1, \quad (x + \beta)(y_{37}) = 1,
\]

\[
ad(y_{34})|_{W_x} = 1_{W_x}, \quad ad(y_{34})|_{W_\beta} = 0, \quad ad(y_{34})|_{W_{x+\beta}} = 1_{W_{x+\beta}},
\]

\[
ad(y_{37})|_{W_x} = 0, \quad ad(y_{37})|_{W_\beta} = 1_{W_\beta}, \quad ad(y_{37})|_{W_{x+\beta}} = 1_{W_{x+\beta}}.
\]

\(^2\) If the characteristic is not two, the Lie algebra \( f_4(C, -) \) is simple (see [9,6]).
where we have defined:

\[ k_1 := 2\zeta + \delta_1 - \epsilon_2 - \eta_1, \]

\[ k_2 := -\zeta - \delta_1 + \eta_1, \]

\[ k_3 := -\zeta - \delta_1 + \epsilon_2, \]

\[ k_4 := \zeta - \epsilon_2 - \eta_1, \]

\[ k_5 := \zeta + \delta_1 - \eta_1, \]

\[ k_6 := \zeta + \delta_1, \]

\[ k_7 := -\zeta + \eta_1, \]

\[ k_8 := -\zeta + \epsilon_2, \]

\[ k_9 := \zeta + \delta_1 - \epsilon_2 - \eta_1, \]

(41)

(42)
The isomorphism, under this structure and using Theorem 2, is: \( \Phi : I \to W \) such that

- \( y_{18} \mapsto y_{28} \)
- \( y_{19} \mapsto y_{46} \)
- \( y_{20} \mapsto y_{48} \)
- \( y_{21} \mapsto y_{43} \)
- \( y_{22} \mapsto y_{41} \)
- \( y_{23} \mapsto y_{39} \)
- \( y_{24} \mapsto y_{31} \)
- \( y_{25} \mapsto y_{52} \)

- \( y_{2} \mapsto y_{35} \)
- \( y_{3} \mapsto y_{49} \)
- \( y_{4} \mapsto y_{27} \)
- \( y_{5} \mapsto y_{50} \)
- \( y_{6} \mapsto y_{30} \)
- \( y_{7} \mapsto y_{36} \)
- \( y_{8} \mapsto y_{32} \)

- \( y_{10} \mapsto y_{18} \)
- \( y_{12} \mapsto y_{18} \)
- \( y_{13} \mapsto y_{44} \)
- \( y_{11} \mapsto y_{35} \)
- \( y_{14} \mapsto y_{28} \)
- \( y_{15} \mapsto y_{46} \)
- \( y_{16} \mapsto y_{18} \)

- \( y_{29} \mapsto y_{10} \)
- \( y_{47} \mapsto y_{11} \)
- \( y_{50} \mapsto y_{12} \)
- \( y_{27} \mapsto y_{13} \)

4. Uncited references

\([1,2,4,5,7]\)

References


