Linearization techniques for singular initial-value problems of ordinary differential equations

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Abstract

Linearization methods for singular initial-value problems in second-order ordinary differential equations are presented. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. The accuracy of these methods is assessed by comparisons with exact and asymptotic solutions of homogeneous and non-homogeneous, linear and nonlinear Lane–Emden equations. It is shown that linearization methods provide accurate solutions even near the singularity or the zeros of the solution. In fact, it is shown that linearization methods provide more accurate solutions than methods based on perturbation methods. It is also shown that the accuracy of these techniques depends on the nonlinearity of the ordinary differential equations and may not be a monotonic function of the step size.

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1. Introduction

Singular initial-value problems in ordinary differential equations occur in several models of mathematical physics and astrophysics [1,2] such as, for
example, the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gas spheres, and theories of thermoionic currents which are modelled by means of the following (generalized) Lane–Emden equation:

\[ y'' + \frac{a}{x} y' = g(x, y), \quad 0 < x < \infty, \quad (1) \]

subject to \( y(0) = A \) and \( y'(0) = B = 0 \), where the prime denotes differentiation with respect to \( x \), and \( a, A \) and \( B \) are constants.

Analytical methods for the solution of Eq. (1) have been based on either series solutions or perturbations techniques. For example, Wazwaz [3] has presented a general framework for obtaining exact and series solutions of Eq. (1) by means of Adomian’s decomposition method [4], while Bender et al. [5] developed a perturbative technique based on the introduction of a small parameter and showed that their technique provides excellent results when applied to nonlinear ordinary differential equations such as the Lane–Emden, Thomas–Fermi, Blasius and Duffing equations.

For linear Lane–Emden equations, i.e., \( g(x, y) = F(x) + G(x)y \), one can use Fröbenius method to determine analytically the series solution of Eq. (1), and this solution may be employed to determine also analytically the solution near the singular point \( x = 0 \). Thereafter, one may use either the analytical solution or a high-order numerical method that employs as initial conditions those determined analytically near the singularity.

Although several methods for singular two-point boundary-value problems in ordinary differential equations such as, for example, the three-point second-order accurate discretizations of Chawla and Katti [6], Chawla et al. [7] and Iyengar and Jain [8] have been developed in the past, the numerical solution of singular initial-value problems has been dealt with by means of quasilinearization techniques [9] and iterated defect correction methods [10]. Quasilinearization techniques are based on the linearization of Eq. (1) and require the solution of a linear ordinary differential equation at each iteration. The convergence rate of this method depends on the initial guess of the solution to Eq. (1), and Mandelzweig and Tabakin [9] have determined general conditions for the quadratic, monotonic and uniform convergence of the quasilinearization method for solving nonlinear ordinary differential equations. The quasilinearization method may be interpreted as a perturbation technique which treats the nonlinear terms as a perturbation about the linear ones, but, unlike perturbation methods, is not based on the existence of a small parameter.

Iterated defect correction methods [10–12] for singular initial-value problems in ordinary differential equations may use an implicit Euler technique whose asymptotic error expansion is employed to accelerate its convergence by means of an iterated defect correction algorithm.

Linearization methods for non-singular initial-value problems (IVP) in ordinary differential equations (odes) have been introduced and applied by the
author and co-workers to a variety of stiff and non-stiff first-order ordinary differential equations [13–16]. Linearization methods for non-singular initial-value problems are based on the piecewise linearization of the nonlinear ode and the analytical solution of the resulting linear ode. Thus, linearization methods provide closed-form solutions in a piecewise fashion and are not iterative; therefore, they do not need a judicious guess to the solution. Furthermore, since these methods are based on the linearization of nonlinear odes, they also provide a means to adapt the step size to the solution. If one is only interested in the values of the dependent variables at discrete values of the independent variable, linearization methods provide explicit nonlinear mappings.

The linearization methods developed previously by the author and co-workers were based on the linearization of the nonlinear odes about the previous step; such a linearization cannot be carried out, however, if there are singular points as, for example, in the Lane–Emden equation. In this paper, we develop linearization methods for the numerical solution of singular initial-value problems of second-order ordinary differential equations, apply them to a variety of homogeneous and non-homogeneous, linear and nonlinear Lane–Emden equations and compare their results with those obtained by means of perturbation methods [5] and Adomain’s decomposition technique [3]. These methods are based on the linearization of Eq. (1), require the solution of linear second-order ordinary differential equations in each time interval, and provide smooth solutions over the whole integration interval. In addition, these techniques are non-iterative and, therefore, do not require an educated guess for the solution as quasilinearization methods do, and do not employ iterative correction algorithms.

The paper has been organized as follows. In Section 2, linearization methods for regular and singular initial-value problems of second-order, nonlinear odes are presented, while, in Section 3, linearization methods are applied to a variety of homogeneous and non-homogeneous, linear and nonlinear Lane–Emden equations. These numerical solutions are compared with available exact or approximate solutions in order to assess the accuracy of the methods presented here. It must be noted that, in this paper, linearization has been performed on the second-order equations rather than on the two-equation system of first-order odes which can be obtained from the second-order one, and have been applied through the whole integration domain. These methods can also be applied to determine the solution near the singularity and, thereafter, one could use a high-order integration technique such as, for example, a fourth-order Runge–Kutta technique; alternatively, a (Fröbenius) series expansion could be used to determine the solution near the singularity and, thereafter, either a high-order method or the linearization techniques presented in this paper could be employed to determine the solution. A summary of the main conclusions puts an end to the paper.
2. Linearization methods

Consider the following nonlinear ode:

\[ y'' = f(x, y, y'), \quad 0 < x < \infty, \]  

subject to

\[ y(0) = y_0, \quad y'(0) = y'_0, \]  

where \( x \) is the independent variable, \( y \) is the dependent variable and \( f(x, y, y') \) is a nonlinear function of \( x, y \) and \( y' \) which may be singular at \( x = 0 \). Consider the interval \((0, L)\) and divide it into a series of subintervals \((x_n, x_{n+1}]\) such that \( x_0 = 0 \). In each subinterval \( f(x, y, y') \) may be linearized as follows. If \( f(x, y, y') \) is regular, \( f \) may be approximated by the first terms of its Taylor series expansion around \((y_n, y_{n}', y'_{n})\), so that Eq. (1) may be approximated by

\[ y'' = f_n(y - y_n) + G_n(y' - y'_{n}) + L_n(x - x_n), \quad x_n < x \leq x_{n+1}, \]  

with

\[ y(x_n) = y_n, \quad y'(x_n) = y'_{n}, \]  

where \( f_n = f(x_n, y_n, y'_{n}), \quad H_n = \frac{\partial f}{\partial y}(x_n, y_n, y'_{n}), \quad G_n = \frac{\partial f}{\partial y'}(x_n, y_n, y'_{n}), \) and \( L_n = \frac{\partial f}{\partial x}(x_n, y_n, y'_{n}) \).

Eq. (3) is a linear ode whose analytical solution may be readily obtained. For example, if \( H_n \neq 0 \) and \( R_n = (G_n/2) + H_n > 0 \), its solution can be written as

\[ y(t) = A_n \exp(\lambda^+_n(x - x_n)) + B_n \exp(\lambda^-_n(x - x_n)) + C_n(x - x_n) + D_n, \]

where \( \lambda^\pm_n = G_n/2 \pm \sqrt{R_n} \), \( C_n = -L_n/H_n \), \( D_n = -(G_nC_n + P_n)/H_n \), \( P_n = f_n - H_n y_n - G_n y'_{n} \) and the constants \( A_n \) and \( B_n \) are determined from Eq. (5) as

\[ A_n = \frac{y'_n - C_n - \lambda^-_n(y_n - D_n)}{\lambda^+_n - \lambda^-_n}, \]

\[ B_n = \frac{\lambda^+_n(y_n - D_n) - (y'_n - C_n)}{\lambda^+_n - \lambda^-_n}, \]

and, therefore,

\[ y_{n+1} = A_n \exp(\lambda^+_n \Delta x_n) + B_n \exp(\lambda^-_n \Delta x_n) + D_n, \]

\[ y'_{n+1} = \lambda^+_n A_n \exp(\lambda^+_n \Delta x_n) + \lambda^-_n B_n \exp(\lambda^-_n \Delta x_n) + C_n, \]

which indicate that \( y_{n+1} \) and \( y'_{n+1} \) are nonlinear maps which depend in a complex manner on \( y_n, y'_{n}, f_n, G_n, H_n \) and \( L_n \), where \( \Delta x = x_{n+1} - x_n \) is the step size which may be varied according to the variation of the solution.
If \( f(x, y, y') \) is singular at \( x_n \), the above derivation is not valid, but \( f \) may be approximated by the first terms of its Taylor series expansion around \((x_{n+1}, y_n, y'_{n})\), so that Eq. (1) may be approximated by

\[
y'' = f_n + H_n (y - y_n) + G_n (y' - y'_{n}) + L_n (x - x_{n+1}), \quad x_n < x \leq x_{n+1},
\]

where \( f_n = f(x_{n+1}, y_n, y'_{n}) \), \( H_n = \frac{\partial f}{\partial y} (x_{n+1}, y_n, y'_{n}) \), \( G_n = \frac{\partial f}{\partial y'} (x_{n+1}, y_n, y'_{n}) \), and \( L_n = \frac{\partial f}{\partial x} (x_{n+1}, y_n, y'_{n}) \), and the solution of the resulting linear ordinary differential equation, for \( H_n \neq 0 \) and \( R_n > 0 \), can be expressed as

\[
y(t) = A_n \exp(\lambda^+_n (x - x_{n+1})) + B_n \exp(\lambda^-_n (x - x_{n+1})) + C_n (x - x_{n+1}) + D_n,
\]

where \( C_n \) and \( D_n \) are given by the same expressions as above and

\[
A_n = \frac{\exp(\lambda^+_n \Delta x_n)}{\lambda^+_n - \lambda^-_n} (y'_n - C_n - \lambda^-_n (y_n + C_n \Delta x_n - D_n)),
\]

\[
B_n = \frac{\exp(\lambda^-_n \Delta x_n)}{\lambda^+_n - \lambda^-_n} (\lambda^+_n (y_n + C_n \Delta x_n - D_n) - (y'_n - C_n)),
\]

and, therefore,

\[
y_{n+1} = A_n + B_n + D_n,
\]

\[
y'_{n+1} = \lambda^+_n A_n + \lambda^-_n B_n + C_n,
\]

which are nonlinear mappings from \((x_n, y_n, y'_{n})\) to \((x_{n+1}, y_{n+1}, y'_{n+1})\).

Analytical solutions to Eqs. (4) and (11) can also be obtained for other conditions than the one considered above. It suffices to mention that these solutions depend on the sign of the radicand, \( R_n \), whether the roots of the characteristic equation are simple or double, etc. Therefore, the analytical solution to Eqs. (4) and (11) may be exponential as in Eqs. (6) and (12), trigonometric or polynomial, and depends on the values of \( G_n \), \( H_n \) and \( L_n \) at \((x_n, y_n, y'_{n})\) for non-singular initial-value problems and those at \((x_{n+1}, y_n, y'_{n})\) for singular initial-value problems. In either case, the linearization method presented in this paper provides analytical solutions in \((x_n, x_{n+1})\) which are continuous with continuous first-order derivative at \( x_n \) and \( x_{n+1} \). If one is only interested in the values at the nodal points (times), \( x_n \), the linearization method provides nonlinear finite difference equations which depend not only on the values of \( y \) and \( y' \) at the previous node (time), but also on the values of \( f, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y'} \) and \( \frac{\partial f}{\partial x} \) at the previous node (time), i.e., the nonlinear finite difference equations carry information of the right-hand side of the ode as well as of its variation with respect to the independent and dependent variables. This information can be used to develop adaptive grid refinement strategies which adapt the step size, \( \Delta x_n \), according to the solution. In this paper, however, a constant step size is used.
The nonlinear mappings or difference equations corresponding to Eqs. (9) and (10) and Eqs. (15) and (16) have been derived by considering that \( f \) is a function of the three variables \( x, y \) and \( y' \). Moreover, since these equations have been obtained by approximating \( f \) by the first terms of its Taylor series expansion, while second- and higher-order terms have been neglected, the local step size \( \Delta x_n \) may be determined from the condition that the second-order terms be much smaller than the first-order ones, or from the condition that \( y_{n+1} \) and \( y'_{n+1} \) do not differ significantly from \( y_n \) and \( y'_n \), respectively.

3. Presentation of results

In order to assess both the applicability and the accuracy of linearization methods, we have applied them to a variety of singular Lane–Emden equations as indicated in the following examples.

Example 1 (Nonlinear, homogeneous Lane–Emden equation). This equation corresponds to Eq. (1) with \( a = 2, g(x, y) = -y^{x}, A = 1 \) and \( B = 0 \), i.e.,

\[
y'' + \frac{2}{x} y' + y^x = 0, \quad y(0) = 1, \quad y'(0) = 0, \tag{17}
\]

and is a nonlinear ode which describes the equilibrium density distribution in a self-gravitating sphere of polytropic isothermal gas, has a singular point at the origin, and is of fundamental importance in the field of stellar structure [1], radiative cooling, modelling of clusters of galaxies, etc. The parameter \( \alpha \) has physical interest in the range \( 0 \leq \alpha \leq 5 \), and Eq. (17) has analytical solutions for \( \alpha = 0, 1 \) and \( 5 \), and has been previously studied by Bender et al. [5] who used perturbation methods and a \((1,1)\)-Padé approximation. In particular, Bender et al. [5] determined the zero of \( y(x) \) here denoted by \( \psi \) asymptotically and found that

\[
\psi = \pi + 0.88527395\delta + 0.24222\delta^2, \tag{18}
\]

for \( \delta = -0.5, 0, 0.5, 1.0 \) and \( 1.5 \) which correspond to \( \alpha = 0, 1, 2, 3 \) and \( 4 \), respectively.

Table 1 shows the zeros of \( y(x) \) corresponding to the exact solution of Eq. (1), the results of the \((1,1)\)-Padé approximation determined by Bender et al. [5] (after correcting some typographical errors), and the numerical solution obtained with the linearization methods presented in this paper. The zeros of the numerical solution were determined by simple linear interpolation of the nodal values at two adjacent points, i.e., at two subsequent times, where the solution changed sign.
Table 1 shows that the linearization methods presented in this paper provide more accurate predictions of the zero of \( y(x) \) than the perturbation technique, especially for large values of \( x \). The accuracy of the linearization methods improves as the step size is decreased from \( 10^{-2} \) to \( 10^{-3} \). The first zero of the solution determined with linearization methods was found to be 6.49170 and 13.87527 for \( x = 3 \) and 4, respectively, for \( \Delta x = 0.1 \), and these values are more accurate than those determined from perturbation methods. These results and those of Table 1 indicate that linearization methods underpredict the zero of the solution for \( \Delta x = 0.1, 0.01 \) and 0.001.

Table 1

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact</th>
<th>(1, 1)-Padé</th>
<th>Numerical ( (\Delta x_n = 10^{-2}) )</th>
<th>Numerical ( (\Delta x_n = 10^{-3}) )</th>
</tr>
</thead>
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<tr>
<td>0</td>
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<td>2.4465</td>
<td>2.44453</td>
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<tr>
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<td>( \pi )</td>
<td>( \pi )</td>
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<tr>
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<tr>
<td>4</td>
<td>14.972</td>
<td>17.967</td>
<td>14.87357</td>
<td>14.96518</td>
</tr>
</tbody>
</table>

Fig. 1 illustrates the values of \( y(x) \) and \( y'(x) \) as well as the absolute errors \( E(y) \equiv |y_e(x) - y(x)| \) and \( E(y') \equiv |y'_e(x) - y'(x)| \) of the solutions obtained with the linearization methods presented in this paper for \( x = 2 \) and two step sizes, where the subscript ‘e’ denotes exact solution. This figure indicates that the absolute errors of both \( y(x) \) and \( y'(x) \) decrease as \( \Delta x \) is decreased. The absolute error of \( y(x) \) increases from its zero value at \( x = 0 \) but its rate of increase decreases as \( x \) increases. At \( x = 4 \), the absolute error of \( y(x) \) for \( \Delta x = 0.001 \) is about one order of magnitude smaller than that for \( \Delta x = 0.01 \). Although similar trends are observed in the absolute errors of \( y'(x) \), it must be noted that the absolute errors of \( y'(x) \) reach an almost constant value for large \( x \) for \( \Delta x = 0.01 \), whereas those for \( \Delta x = 0.001 \) show a smaller rate of increase than the absolute errors of \( y(x) \) for \( x > 2 \) and exhibit a minimum at about \( x = 2 \).

The results presented in Fig. 1 indicate that if \( y_e(x_n) - y_n = C\Delta x^\beta \), then \( \beta = 1 \), whereas, for non-singular initial-value problems, a Taylor series expansion of Eqs. (9) and (15) indicates that the accuracy of these equations is \( \Delta x^2 \); thus, the accuracy of the linearization method drops when these methods are applied to singular IVP. A similar result has been found whenever the trapezoidal or box methods are applied to singular initial-value problems because the asymptotic expansion of the global error of these methods breaks down due to the singularity; in fact, the remainder terms of these methods cannot be estimated at the necessary level of accuracy. It is for this reason that
iterated defect correction methods have been employed with the implicit Euler technique when dealing with singular initial-value problems [10–12].

The results presented in Table 2 indicate that, for $x = 2$, despite the singularity at $x = 0$, the absolute errors of the linearization methods are nearly $10^{-5}$ and $10^{-3}$ for $y(0.01)$ and $y'(0.01)$, respectively, for $\Delta x = 0.01$, and almost $10^{-6}$ and $10^{-4}$, respectively, for $\Delta x = 0.001$. Moreover, $y(\Delta x)$ and $y'(\Delta x)$ are nearly equal to $10^{-7}$ and $10^{-4}$ for $\Delta x = 0.001$. These results indicate that for Example 1, the accuracy of $y(0.01)$ and $y'(0.01)$ increases as $\Delta x$ is decreased.

Mandelzweig and Tabakin [9] employed a quasilinearization method after introducing the change of variable $u = y/x$ to solve Eq. (17) and found that the difference between the exact solution and the eighth iteration of their quasilinearization method was less than $10^{-11}$ for $0 \leq x \leq 10$ and $x = 4$. For $x = 0, 1$ and $5$, the difference between the exact solution and the numerical results of the linearization method presented here are less than $10^{-9}$ at $x = 10$ for $\Delta x_n = 0.00001$, whereas this difference is less than $10^{-11}$ at $x = 10$ for $\Delta x_n = 0.000001$, and is of the same order of magnitude as that corresponding to the eighth iteration of the quasilinearization method [9].

![Graphs](ftn57.1)

Fig. 1. Solution $y(x)$ (top left) and $y'(x)$ (top right), and absolute errors $E(y) = |y_e(x) - y(x)|$ (bottom left) and $E(y_p) = |y'_e(x) - y'(x)|$ (bottom right) for Example 1. Solid and dashed lines correspond to $\Delta x_n = 0.01$ and 0.001, respectively.
Examples 2–7. The smallest value of $x \equiv D$ and $0.005$, respectively, except for Example 1 where they denote calculations performed with paper corresponding to the following linear, non-homogeneous Lane–Emden equation:

$$y'' + \frac{2}{x} y' + y = 6 + 12x + x^2 + x^3$$

subject to $y(0) = 0$ and $y'(0) = 0$, which has the following analytical solution:

$$y_c(x) = x^2 + x^3.$$  

The absolute errors of $y$ and $y'$ first increase from $x = 0$ and reach a maximum near this location; they then decrease and then increase. The absolute errors of $y$ and $y'$ reach almost constant values for $x > 7$ as illustrated in Fig. 2; these constant values are on the order of $10^{-2.5}$. Fig. 2 also shows the relative errors of $y(x)$ and $y'(x)$ of the solutions, i.e., $\text{RE}(y) = |y_c(x) - y(x)| / |y_c(x)|$ and $\text{RE}(y') = |y'_c(x) - y'(x)| / |y'_c(x)|$, obtained with the linearization methods presented in this paper, and indicates that the relative errors decrease from a large value at $x = 0$ where $y(0) = 0$ to about $10^{-5}$ for $x \geq 5$. This decrease is non-monotonic, and the relative errors are not monotonic functions of the step size. In fact, a step size $\Delta x = 0.005$ may result in larger relative errors of $y(x)$ for small $x$ than $\Delta x = 0.01$. Moreover, the relative errors of linearization methods tend to reach a constant value at large $x$ for Eq. (19).

The results presented in Table 2 indicate that the absolute errors of the linearization methods are nearly $10^{-3.8}$ and $10^{-2.2}$ for $y(0.01)$ and $y'(0.01)$, respectively, for $\Delta x = 0.01$, and almost $10^{-2.4}$ and $10^{-2.7}$, respectively, for $\Delta x = 0.005$. Moreover, $y(\Delta x)$ and $y'(\Delta x)$ are nearly equal to $10^{-4.6}$ and $10^{-2.5}$ for $\Delta x = 0.005$. These results indicate that the accuracy of $y(0.01)$ degrades whereas that of $y'(0.01)$ increases as $\Delta x$ is decreased. Table 2 and Fig. 2 also

<table>
<thead>
<tr>
<th>Example</th>
<th>$\log E_1(y_1)$</th>
<th>$\log E_1(y_1')$</th>
<th>$\log E_2(y_1)$</th>
<th>$\log E_2(y_1')$</th>
<th>$\log E_2(y_1)$</th>
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</tr>
</tbody>
</table>

$E_1$ and $E_2$ denote the absolute value of the errors incurred in calculations performed with $\Delta x = 0.01$ and $0.005$, respectively, except for Example 1 where they denote calculations performed with $\Delta x = 0.01$ and $0.001$, respectively. $y_1$ and $y_1'$ denote the values of $y$ and $y'$ obtained numerically at $x = 0.01$, whereas the subscript ‘$\ast$’ corresponds to the first step of calculations performed with the smallest value of $\Delta x$ employed in each example, i.e., $\Delta x = 0.001$ for Example 1 and 0.005 for Examples 2–7.
show that the absolute errors first increase from their zero value at $x = 0$ and reach a relative maximum near the initial value of the independent variable.

**Example 3** (*Linear, homogeneous Lane–Emden equation*). This example corresponds to the following linear, homogeneous Lane–Emden equation:

$$y'' + \frac{2}{x}y' = 2(2x^2 + 3)y$$

subject to $y(0) = 1$ and $y'(0) = 0$, which has the following analytical solution:

$$y_e(x) = e^{x^2}.$$  \hfill (21)

For this example, the results presented in Fig. 3 indicate that the relative errors of $y(x)$ increase from very low values at $x = 0$ in a monotonic fashion, while those of $y'(x)$ first decrease and then increase. In addition, the relative errors for $\Delta x = 0.005$ are about one order of magnitude smaller than those for $y'(x)$ for $x \geq 1$. Fig. 3 also shows that the absolute errors of both $y$ and $y'$ in-
crease as \( x \) increases in accord with the exact solution which grows exponentially.

The results presented in Table 2 indicate that the absolute errors of the linearization methods are nearly \( 10^{-4.2} \) and \( 10^{-2.2} \) for \( y(0.01) \) and \( y'(0.01) \), respectively, for \( \Delta x = 0.01 \), and almost \( 10^{-4.7} \) and \( 10^{-3.7} \), respectively, for \( \Delta x = 0.001 \). Moreover, \( y(\Delta x) \) and \( y'(\Delta x) \) are nearly equal to \( 10^{-4.8} \) and \( 10^{-2.5} \) for \( \Delta x = 0.001 \). Thus, for Example 3, the accuracy of \( y(0.01) \) and \( y'(0.01) \) increases as \( \Delta x \) is decreased.

**Example 4** (*Nonlinear, homogeneous Lane–Emden equation*). This example corresponds to the following nonlinear, homogeneous Lane–Emden equation:

\[
y'' + \frac{2}{x}y' + 4(2e^y + e^x) = 0
\]

subject to \( y(0) = 0 \) and \( y'(0) = 0 \), which has the following analytical solution:

\[
y_e(x) = -2\ln(1 + x^2).
\]
The absolute errors of both $y$ and $y'$ first increase from $x = 0$ and reach a maximum near this location as illustrated in Fig. 4; they then decrease before rising sharply and then decreasing slowly. The absolute errors of both $y$ and $y'$ at $x = 10$ are on the order of $10^{-4}$. Fig. 4 also shows that the relative errors of $y(x)$ and $y'(x)$ decrease from a large value at $x = 0$ where $y(0) = 0$ to about $10^{-5}$ and $10^{-4}$, respectively, for $x \geq 8$. This decrease is non-monotonic, and the relative errors are, in this example, monotonic functions of the step size. Fig. 4 also shows that the relative error of $y(x)$ decreases as $x$ is increased, whereas that of $y'(x)$ reaches an almost constant value for large values of $x$.

Table 2 indicates that the accuracy of $y(x)$ and $y'(x)$ increases as $\Delta x$ is decreased, and the accuracy of $y'$ at $x = 0.01$ is higher than that at 0.005 for $\Delta x = 0.005$, whereas just the opposite is true for $y$.

**Example 5 (Nonlinear, homogeneous Lane–Emden equation).** This example corresponds to the following nonlinear, homogeneous Lane–Emden equation:

$$y'' + \frac{2}{x} y' - 6y = 4y \ln y$$

(25)
subject to $y(0) = 1$ and $y'(0) = 0$, which has the following analytical solution:

$$y_e(x) = e^{x^2}. \quad (26)$$

Fig. 5 shows that the relative errors of $y(x)$ increase from very low values in a monotonic fashion for small $x$ and tend to constant values for $x \geq 7$, while those of $y'(x)$ first decrease and then increase, and tend to a constant value. In addition, the relative errors for $\Delta x = 0.005$ are less than one order of magnitude smaller than those for $y'(x)$ for $x \geq 1$, and the difference between the relative errors corresponding to $\Delta x = 0.005$ and 0.01 decreases as $x$ increases. Fig. 5 also shows that the absolute errors of both $y$ and $y'$ increase as $x$ increases in accord with the exact solution which grows exponentially.

The results presented in Table 2 indicate that the accuracy of $y(0.01)$ and $y'(0.01)$ increases as $\Delta x$ is decreased, and the accuracy of $y'$ at $x = 0.01$ is higher than that at 0.005 for $\Delta x = 0.005$, whereas just the opposite is true for $y$.

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Fig. 5. Absolute errors $E(y) = |y_e(x) - y(x)|$ (top left), $E(y') = |y'_e(x) - y'(x)|$ (top right), and relative errors $RE(y) = |y(x) - y(x)|/|y_e(x)|$ (bottom left) and $E(y') = |y'(x) - y'(x)|/|y'_e(x)|$ (bottom right) for Example 5. Solid and dashed lines correspond to $\Delta x_n = 0.01$ and 0.005, respectively.
Example 6 (Linear, non-homogeneous Lane–Emden equation). This example corresponds to the following linear, homogeneous Lane–Emden equation:

\[ y'' + \frac{8}{x} y' + xy = x^5 - x^4 + 44x^2 - 30x \]

subject to \( y(0) = 0 \) and \( y'(0) = 0 \), which has the following analytical solution:

\[ y_e(x) = x^4 - x^3. \]

The exact solution has zeros at \( x = 0 \) and 1, while \( y'_e(x) \) is nil at \( x = 0 \) and \( 3/4 \). The results presented in Fig. 6 indicate that the relative error of the numerical solution is largest at \( x = 1 \), whereas that of \( y' \) is largest at \( 3/4 \). This figure also shows that the relative errors for \( \Delta x = 0.005 \) are smaller than those for \( 0.01 \), and that the numerical solution exhibits a small relative error at \( 3/4 \); this error is due to the fact that \( y'_e(3/4) = 0 \). Fig. 6 also indicates that the relative error for \( \Delta x = 0.01 \) and 0.005 decrease slowly as \( x \) increases for \( x > 1 \).

The absolute errors presented in Fig. 6 increase from their lowest value at \( x = 0 \) and then decrease before increasing as \( x \) increases. The absolute errors of \( y \) increase monotonically for \( x \geq 3/4 \) regardless of the step size, whereas those of \( y' \) increase monotonically for \( x \geq 3/4 \) and \( x \geq 0.5 \) for \( \Delta x = 0.01 \) and 0.005, respectively, and the absolute errors of both \( y \) and \( y' \) are about \( 10^{-1} \) at \( x = 2 \).

Table 2 indicates that the accuracy of \( y(0.01) \) and \( y'(0.01) \) increases as \( \Delta x \) is decreased, and the accuracy of both \( y \) and \( y' \) at \( x = 0.01 \) is lower than that at \( 0.005 \) for \( \Delta x = 0.005 \). Table 2 also indicates that the absolute errors of \( y'(0.01) \) are nearly identical for \( \Delta x = 0.01 \) and 0.005.

Example 7 (Nonlinear, homogeneous Lane–Emden equation). This example corresponds to the following nonlinear, homogeneous Lane–Emden equation:

\[ y'' + \frac{6}{x} y' + 14y = -4y \ln y \]

subject to \( y(0) = 1 \) and \( y'(0) = 0 \), which has the following analytical solution

\[ y_e(x) = e^{-x^2}. \]

Fig. 7 indicates that the absolute errors of both \( y \) and \( y' \) increase from their lowest value at \( x = 0 \) and then decrease in a non-monotonic fashion. At \( x = 2 \), the absolute errors of \( y \) and \( y' \) are about \( 10^{-4.5} \). The relative errors of the numerical solution presented in Fig. 7 first increase up to about \( x = 0.1 \), then slowly decrease, before dropping rapidly to very small values at \( x = 1.5 \), and then increase sharply. On the other hand, the relative errors of the first derivative of the solution decrease and reach their smallest value at about \( x = 0.34 \); beyond this location, they increase and then decrease slowly as \( x \) increases.
The results presented in Table 2 indicate that the accuracy of \( y(0.01) \) and \( y'(0.01) \) increases as \( \Delta x \) is decreased, and the accuracy of both \( y \) and \( y' \) at \( x = 0.01 \) is higher than that at 0.005 for \( \Delta x = 0.005 \).

In view of the results presented in Figs. 1–7 and Tables 1 and 2, it may be stated that linearization methods predict accurately the solution of singular initial-value problems of second-order ordinary differential equations, even near the initial singularity. They also predict accurately the zeros of both the solution and its first-order derivative. These methods provide piecewise analytical solutions which are globally continuous and differentiable. However, these methods suffer from a drop in the order of accuracy in some singular IVP and their accuracy depends on the nonlinearity and non-homogeneous terms of the ordinary differential equation as well as the step size. Such a drop in accuracy may be substantially reduced by employing (Fröbenius) series expansions and determining the solution analytically but approximately near the singular point; thereafter, there are no singularities and the method presented in this paper or that for regular points (cf. Eqs. (9) and (10)) which are second-order accurate or a high-order numerical algorithm could be used to determine the solution.

Fig. 6. Absolute errors \( E(y) = |y(x) - y(x)| \) (top left), \( E(y') = |y'(x) - y'(x)| \) (top right), and relative errors \( RE(y) = |y(x) - y(x)|/|y(x)| \) (bottom left) and \( E(y') = |y'(x) - y'(x)|/|y'(x)| \) (bottom right) for Example 6. Solid and dashed lines correspond to \( \Delta x_n = 0.01 \) and 0.005, respectively.
For very small step sizes, the exponential or trigonometric functions which correspond to the homogeneous solutions may be subject to catastrophic cancellations and large round-off errors. On the other hand, large time steps may result in overflow of exponential terms. The first problem can be somewhat avoided by expanding in Taylor series the exponential or trigonometric functions and retaining a reasonable number of terms, whereas the latter can be avoided by employing smaller step sizes whose magnitude may be controlled by the evolution of the solution and the need to avoid large exponents.

4. Conclusions

Linearization methods for singular initial-value problems of second-order ordinary differential equations have been developed and applied to a variety of homogeneous and non-homogeneous, linear and nonlinear Lane–Emden equations. These methods are based on the linearization of the ordinary differential equation with respect to both the dependent and the

Fig. 7. Absolute errors $E(y) = |y_e(x) - y(x)|$ (top left) and $E(y') = |y'_e(x) - y'(x)|$ (top right), and relative errors $RE(y) = |y_e(x) - y(x)|/|y_e(x)|$ (bottom left) and $E(y') = |y'_e(x) - y'(x)|/|y'_e(x)|$ (bottom right) for Example 7. Solid and dashed lines correspond to $\Delta x_n = 0.01$ and 0.005, respectively.
independent variables in subintervals, provide analytical solutions within each subinterval, and globally smooth solutions in the whole interval, because the solution and its first-order derivative are required to be continuous at the ends of each subinterval. If one is only interested in the solution at certain nodes, the linearization methods developed in this paper for second-order ordinary differential equations provide nonlinear mappings which depend in a complex manner on the values of the solution and its first-order derivative, and the partial derivatives of the nonlinear function with respect to the independent and dependent variables at the previous node (time).

The linearization methods presented in this paper do not need an initial guess, are not iterative, and may use variable step sizes according to the evolution of the solution, whereas quasilinearization techniques do require an initial guess and are iterative. For non-singular initial-value problems, linearization methods are robust, stable and accurate techniques whose accuracy is comparable to that of quasilinearization techniques and is not a strong function of either the linearization point or the linearization with respect to the independent variable.

For odes with singular points such as the Lane–Emden equation, it has been found that linearization methods provide more accurate results than those obtained by means of perturbation methods even for relatively large and equal step sizes. These methods have also been found to predict accurately the first zero of the solution of several homogeneous and non-homogeneous, linear and nonlinear Lane–Emden equations, and the exponentially growing solutions of several singular initial-value problems. Only for one of the examples considered in this paper, i.e., Example 6, it has been found that the accuracy of linearization methods degrades for large values of the independent variables, but these methods do predict accurately the behaviour of the solution near the singularity and the first zero of the solution. However, the accuracy of these methods for Example 6 is substantially improved by employing the formulation presented in this paper for regular problems or high-order numerical integrators for odes beyond the first zero of the solution.

The linearization methods presented in this paper avoid the initial singularity and have been used throughout the whole integration domain. Alternatively, a (Fröbenius) series expansion could be used to determine analytically but approximately the solution near the singularity and, thereafter, applying the time linearization method presented in this paper or that for ordinary differential equations with regular points both of which are second-order accurate.

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References