Exponential methods for one-dimensional reaction–diffusion equations

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Abstract

Non-iterative exponential time-linearization and iterative exponential quasilinearization techniques for one-dimensional reaction–diffusion equations based on the discretization of the time derivative, the freezing of the coefficients of the resulting linear ordinary differential equations and the piecewise analytical solution of these ordinary differential equations, are presented and applied to the Nagumo and Fisher equations. These techniques yield three-point finite difference expressions that depend in an exponential manner on either the diffusion, reaction and transient terms or the diffusion and reaction terms. It is shown that first-order accurate in time, exponential time-linearization and exponential quasilinearization methods which account for reaction and diffusion processes in the (ordinary) differential operator are more robust than standard second-order accurate techniques even for very large time steps and grid sizes, and yield results in excellent accord with those obtained by means of non-standard finite difference schemes and nodal integral formulations for the Nagumo and Fisher equations. Unlike non-standard finite difference methods, the exponential schemes presented in this paper do not require any knowledge of the exact solution of the differential equation and can be easily derived in a rather systematic manner. Compared with nodal integral formulations, the exponential techniques presented here only require the solution of second-order ordinary differential equations.

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1. Introduction

This paper deals with the numerical solution of one-dimensional, nonlinear reaction–diffusion equations, i.e.,

\[
\frac{\partial u}{\partial t} = d \frac{\partial^2 u}{\partial x^2} + S(u), \quad a < x < b, \quad t > 0,
\]

by means of exponential methods, where \( t \) is time, \( x \) the spatial coordinate, \( d \) is the (constant) diffusion coefficient, and \( S \) is the nonlinear reaction term.

Eq. (1) appears in numerous applications such as heat and mass transfer, combustion, biology, etc., and there has been a wide variety of numerical methods, e.g., finite difference techniques, finite element methods, spectral techniques, adaptive and non-adaptive algorithms, etc., which have been developed for its numerical solution [1,2]. Amongst the most recent numerical techniques for Eq. (1), it is worth mentioning non-standard finite difference methods [3,4], hybrid boundary integral procedures [5], the nodal integral scheme [8] and piecewise hybrid analytical–numerical algorithms [9]. Non-standard finite difference methods include exact finite difference and best finite difference schemes. A finite difference scheme is called an exact finite difference one if the solution for all the step sizes at all grid points is equal to the exact analytical solution of the differential equation at the corresponding grid points. Therefore, one should expect that only those differential equations which have a closed-form exact solution will lead to exact finite difference schemes.

A finite difference scheme is called a best finite difference one if the important properties of its solution correspond exactly to the related properties of the solution of the differential equation for all values of the step size [6,7]. Some of these important properties include conservation, exact solutions, preservation of fixed points, stability, etc.

Nodal integral schemes [8] are based on the division of the space–time domain into layers and space-averaged/time-dependent and time-averaged/space-dependent densities and yield second-order ordinary differential equations (in space) for the time-step averaged, space-dependent, local densities, and first-order ordinary differential equations (in time) for the space-averaged, time-dependent, local densities. By expanding the source terms that appear in these ordinary differential equations in terms of, say, truncated Legendre polynomials, analytical solutions of these equations can be obtained. These analytical solutions are trigonometric and exponential for the time-averaged and space-averaged densities, respectively, and provide three-point schemes. Unlike
finite difference methods that provide nodal values, the discrete values of nodal integral schemes are averaged quantities, and these schemes satisfy the original partial differential equation in an integral fashion, are similar to the best finite difference methods and capture the shape and speed of one-dimensional propagating fronts accurately. By way of contrast, the best finite difference scheme is stable for large grid sizes, but at the cost of inaccuracies in the front propagation speed [8]. In addition, the nodal integral approach is more systematic than the rather ad hoc development of non-standard finite difference methods.

In this paper, exponential methods for the numerical solution of Eq. (1) are developed and applied to the Nagumo and Fisher equations. These methods are based on the discretization of the time derivative in Eq. (1) which yields a linear ordinary differential equation at each time level and the piecewise analytical solution of this ordinary differential equation. As a consequence, the exponential methods presented in this paper are hybrid techniques, for they use piecewise analytical solutions and result in three-point finite difference equations for the nodal values, account for either the reaction, diffusion and transient terms or the reaction and diffusion terms in the piecewise linear (ordinary) differential operators, and yield piecewise analytical solutions of the exponential type. In addition, these methods do not require a knowledge of the exact solution of Eq. (1), their finite difference equations are non-standard, their development is more systematic than those of exact finite difference and best finite difference schemes [3,4,6,7], and can be easily applied to multidimensional reaction–diffusion problems by employing factorization techniques that reduce a multidimensional problem to a sequence of one-dimensional ones [10,11].

The paper has been organized as follows. In Section 2, we study the properties of the time linearization method presented in this paper for the diffusionless Nagumo equation, and show that this method preserves the fixed points and stability of the original (first-order) Nagumo equation. Exponential time linearization and exponential quasilinearization techniques based on the inclusion of either the reaction, diffusion and transient terms or the reaction and diffusion terms in the (ordinary differential) operators that result upon time discretization of Eq. (1) are presented in Section 3. The exponential techniques are applied to the Nagumo and Fisher equations in Section 4, where they are compared with standard second-order accurate time linearization and quasilinearization methods [1,2]. A summary of the main conclusions puts an end to the paper.

2. Diffusionless Nagumo’s equation

In this section, we consider the case \(d = 0\), i.e., homogeneous reaction, and \(S(u) = \alpha u - \beta u^3\), i.e., the Nagumo reaction rate, and present a summary of analytical results which are used as a reference to assess the nonlinear dynamical properties of the time linearization methods presented in this paper.
2.1. Analytical results

When $d = 0$ with $S(u) = xu - \beta u^3$, Eq. (1) reduces to the following ordinary differential equation:

$$\frac{du}{dt} = xu - \beta u^3, \quad t > 0,$$

whose equilibrium points are $u^* = 0$ and $u_{\pm} = \pm \sqrt[3]{\frac{a}{\beta}} \sqrt[3]{\frac{a}{\beta}}$. Since the Jacobian $J(u) = x - 3\beta u^2$, the eigenvalue associated with $u^*$ is $\lambda^* = x$ which is zero when $x = 0$. Analogously, $J(u_{\pm}) = -2x$. Therefore, bifurcation occurs for $x = 0$, and $u^* = 0$ is locally asymptotically stable for $x < 0$ and unstable for $x > 0$; $u_{\pm}$ is locally asymptotically stable for $x > 0$. The bifurcation at $x = 0$ is of the pitchfork type, and it is easy to show that $u^* = 0$ is globally asymptotically stable if $a < 0$ and $\beta > 0$, and globally asymptotically stable in the interval $u_0 \in \left(-\sqrt[3]{\frac{a}{\beta}}, \sqrt[3]{\frac{a}{\beta}}\right)$ if $x < 0$ and $\beta < 0$. The basins of attraction of $u_+$ and $u_-$ are $u_0 \in (0, \infty)$ and $u_0 \in (-\infty, 0)$, respectively.

2.2. Time-linearized method

Eq. (2) can be approximated in the interval $(t^n, t^{n+1}]$ by

$$\frac{du}{dt} = S^n + J^n(u - u^n), \quad t^n < t \leq t^{n+1},$$

where $S = xu - \beta u^3$ and $J = \frac{dS}{du}$, $n$ denotes the $n$th time level and $\Delta t^n \equiv t^{n+1} - t^n$.

The solution of Eq. (3) subject to $u(t^n) = u^n$ can be determined analytically and is of exponential type if $J^n \neq 0$ and a linear function of $t$ if $J^n = 0$. Moreover, Eq. (3) yields

$$u^{n+1} = g(u^n),$$

where

$$g(u) = u \frac{(x - \beta u^2)e^{(x-3\beta u^2)\Delta t}}{x - 3\beta u^2},$$

if $J_n \neq 0$, and

$$g(u) = u(1 + 2\beta u^2 \Delta t),$$

if $J^n = 0$, i.e., if $u^n = \pm \sqrt[3]{\frac{x}{3\beta}}$, and, therefore, $u^{n+1} = \pm \sqrt[3]{\frac{x}{3\beta}(1 + \frac{2}{3} x \Delta t)}$.

The fixed points of Eq. (4), i.e., $u = g(u)$, for $J_n \neq 0$ are $u = 0$ and $u = \pm \sqrt[3]{\frac{x}{\beta}}$.

Since $g'(0) = e^{2\Delta t}$, $u = 0$ is stable if $|g'(0)| < 1$, i.e., if $x < 0$; moreover, $g'(\pm \sqrt[3]{\frac{x}{\beta}}) = e^{-2x \Delta t}$ and, therefore, $u = \pm \sqrt[3]{\frac{x}{\beta}}$ is stable if $x > 0$. Thus, the
time-linearized method preserves the fixed points and the linear stability associated with the original Nagumo equation in the absence of diffusion.

3. Exponential methods for one-dimensional reaction–diffusion equations

Eq. (1) can be transformed into a linear ordinary differential equation by discretizing the time derivative by means of a \( h \)-method and linearizing the nonlinear source term, \( S(u) \), with respect to either the previous time level or the previous iteration. If the linearization is performed with respect to the previous time level, one obtains

\[
\frac{u - u^n}{\Delta t} = \theta d \frac{d^2 u}{dx^2} + (1 - \theta) d \frac{d^2 u^n}{dx^2} + S^n + \theta J^n(u - u^n), \quad t^n < t \leq t^{n+1},
\]

where \( u \equiv u^{n+1} \), and Eq. (7) corresponds to a (non-iterative) time linearization method, whereas if the linearization is performed with respect to the previous iteration, one obtains

\[
\frac{u - u^n}{\Delta t} = \theta d \frac{d^2 u}{dx^2} + (1 - \theta) d \frac{d^2 u^n}{dx^2} + (1 - \theta)S^n + \theta S^k + \theta J^k(u - u^k), \quad t^n < t \leq t^{n+1},
\]

where \( u \equiv u^{k+1} \), and Eq. (8) corresponds to an (iterative) quasilinear technique which reaches convergence when a user’s specified convergence criterion is satisfied. In this paper, convergence of quasilinear methods was achieved whenever

\[
E^2 = \frac{1}{NP} \sum_{i=1}^{NP} (u_{i}^{k+1} - u_{i}^{k})^2 \leq 10^{-16},
\]

where \( u_{i} = u(x_{i}, t^{n+1}) \) and \( NP \) denotes the number of grid points.

Eqs. (7) and (8) cannot, in general, be solved in an analytical manner because of the spatial dependence of \( u^n, S^n \) and \( J^n \) or \( u^k, S^k \) and \( J^k \) for the time-linearized or iterative quasilinear techniques, respectively. However, if the interval \([a, b]\) is divided into \( N \) equally-spaced intervals and, in the interval \((x_{i-1}, x_{i+1})\), the spatially dependent coefficients of Eqs. (7) and (8) are approximated by their values at \( x_{i} \), then one obtains the following (piecewise) linear constant-coefficients ordinary differential equation:

\[
\frac{u - u_{i}^n}{\Delta t} = \theta d \frac{d^2 u}{dx^2} + (1 - \theta) d \frac{d^2 u_{i}^n}{dx^2} + S_{i}^n + \theta J_{i}^n(u - u_{i}^n), \quad x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1},
\]

\[
\frac{u - u_{i}^n}{\Delta t} = \theta d \frac{d^2 u}{dx^2} + (1 - \theta) d \frac{d^2 u_{i}^n}{dx^2} + (1 - \theta)S_{i}^n + \theta S_{i}^k + \theta J_{i}^k(u - u_{i}^k), \quad x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1},
\]

respectively.
Eqs. (10) and (11) can be solved in closed form in \((x_{i-1}, x_{i+1})\) subject to the following conditions:

\[ u(x_{i-1}) = u_{i-1}, \quad u(x_i) = u_i, \quad u(x_{i+1}) = u_{i+1}, \]  

(12)

and yield exponential solutions in \((x_{i-1}, x_{i+1})\) which are analytical in that interval and continuous everywhere.

Since Eqs. (10) and (11) are very similar, we shall only present in detail exponential methods for Eq. (10) in the following subsections.

### 3.1. Time-linearized full exponential techniques

Time-linearized full exponential techniques are obtained from the piecewise analytical solution of Eq. (10) rewritten as

\[
0 \frac{d^2u}{dx^2} + \left( \partial J^n_i - \frac{1}{\Delta t} \right) u = T^n_i \equiv -(1 - \theta) \frac{d^2u^n_i}{dx^2} - S^n_i + u^n_i \left( \partial J^n_i - \frac{1}{\Delta t} \right),
\]

\(x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1},\)

(13)

which accounts for diffusion, reaction and transient effects in the differential operator i.e., the left-hand side of Eq. (13).

The solution of Eq. (13) depends on the sign of \(H^n_i \equiv \partial J^n_i - \frac{1}{\Delta t} \neq 0\). For example, if \(H^n_i = 0\), the solution of Eq. (13) subject to Eq. (12) can be readily obtained and one can easily deduce the following three-point finite difference expression:

\[ u_{i-1} - 2\cosh(\lambda_i h)u_i + u_{i+1} = 2\gamma_i(1 - \cosh(\lambda_i h)), \quad t^n < t \leq t^{n+1},\]

(14)

where \(\gamma_i = \frac{T^n_i}{h^2}\), and the constants \(A_i\) and \(B_i\) can be determined from Eq. (12) which yield the following three-point finite difference expression:

\[ u_{i-1} - 2u_i + u_{i+1} = T^n_i h^2, \quad t^n < t \leq t^{n+1}.\]

(15)

If \(H_i = 0\), the solution of Eq. (13) subject to Eq. (12) can be readily obtained and one can easily deduce the following finite difference expression:

\[ u_{i-1} - 2\cos(\omega_i h)u_i + u_{i+1} = 2\gamma_i(1 - \cos(\omega_i h)), \quad t^n < t \leq t^{n+1}.\]

(16)

If \(H_i = \theta d\omega_i^2 > 0\), the solution of Eq. (13) and the conditions of Eq. (12) allow one to obtain

\[ u_{i-1} - 2\cos(\omega_i h)u_i + u_{i+1} = 2\gamma_i(1 - \cos(\omega_i h)), \quad t^n < t \leq t^{n+1}.\]

(17)

Hereon, the values \(\theta = 1\) and \(\frac{1}{2}\) in Eqs. (15), (16) and (17) correspond to the time linearization methods referred to as TFET and TFET2, respectively. In the latter, the explicit term \(\frac{d^2u^n_i}{dx^2}\) in \(T^n_i\) is approximated by \(\frac{1}{h^2}\delta^2u^n_i\) where \(\delta^2u_i = u_{i-1} - 2u_i + u_{i+1}\), although this term could be evaluated analytically from either the initial condition for the first time step or from the piecewise exponential solutions given by Eqs. (14)–(17) for \(n > 1\).
The main advantage of the full exponential techniques presented above is that they account for the reaction, diffusion and transient terms in determining the homogeneous solution of Eq. (13). However, this technique also has the inconvenient that the time step cannot be chosen carelessly because, if \( \Delta t \ll \frac{1}{|J_i|} \), then the reaction terms do not affect much the values of \( \lambda_i \) and \( \omega_i \) although they do affect the particular solution of Eq. (13) through \( T^n \). In addition, part of the transient term affects the ordinary differential operator and, therefore, the homogeneous solution, whereas other part affects the particular solution. These drawbacks as well as the results of many numerical experiments are entirely eliminated with the exponential techniques presented in the third and fourth subsections.

### 3.2. Quasilinear full exponential techniques

These techniques correspond to the iterative solution of

\[
\theta d^2 u \frac{d^2 u}{dx^2} + \left( \theta J_i^k - \frac{1}{\Delta t} \right) u = T_i^k \equiv -(1 - \theta)d^2 u_i^m \frac{d^2 u_i^m}{dx^2} - (1 - \theta)S_i^m - \theta S_i^k + \theta J_i^k u_i^k - u_i^m, \\
x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1},
\]

until a user’s specified convergence criterion is satisfied (cf. Eq. (9)), and have analogous solutions to those reported in the previous section, i.e., Eqs. (14)–(17).

Hereon, the values \( \theta = 1 \) and \( \frac{1}{2} \) in Eq. (18) correspond to the quasilinear methods referred to as \( \text{QFET} \) and \( \text{QFET2} \), respectively. In the latter, the explicit term \( d^2 u_i^m \frac{d^2 u_i^m}{dx^2} \) in \( T_i^k \) is approximated in the same manner as in the \( \text{TFET2} \) method. The exponential quasilinear \( \text{QFET} \) and \( \text{QFET2} \) described in this section have the same advantages and limitations as the exponential time linearization methods presented in the previous section.

### 3.3. Time-linearized exponential techniques

The time-linearized exponential techniques presented in this section consider the following differential operator (cf. Eq. (13)):

\[
\theta d^2 u \frac{d^2 u}{dx^2} + \theta J_i^m u = Q_i^m \equiv -(1 - \theta)d^2 u_i^m \frac{d^2 u_i^m}{dx^2} - S_i^m + \theta J_i^m u_i^m + \frac{u_i - u_i^n}{\Delta t}, \\
x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1},
\]

which only accounts for reaction and diffusion processes in the differential operator and whose solutions depend on the sign of \( J_i^m \) as follows:
The solution of Eq. (19) and conditions of Eq. (12) yield
\[ u_{i-1} - \frac{2 + (J^n_i \Delta t - 1)(e^{J^n_i} + e^{-J^n_i})}{J^n_i \Delta t} u_i + u_{i+1} = \frac{2 - (e^{J^n_i} + e^{-J^n_i})}{J^n_i \Delta t} (-u^n_i + \Delta t(J^n_i u^n_i - S^n_i)), \quad t^n < t \leq t^{n+1}, \] (20)
if \( J^n_i = -d \dot{x}_i^2 < 0, \)
\[ u_{i-1} - \left( 2 + \frac{\dot{x}_i^2}{\Delta t} \right) u_i + u_{i+1} = -\frac{\dot{x}_i^2}{\Delta t} (u^n_i + \Delta t S^n_i), \quad t^n < t \leq t^{n+1}, \] (21)
if \( J^n_i = 0, \) and
\[ u_{i-1} - \frac{1 + (J^n_i \Delta t - 1) \cos(\omega \Delta t)}{J^n_i \Delta t} u_i + u_{i+1} = \frac{1 - \cos(\omega \Delta t)}{J^n_i \Delta t} (-u^n_i + \Delta t(J^n_i u^n_i - S^n_i)), \quad t^n < t \leq t^{n+1}, \] (22)
if \( J^n_i = d \omega^2 \dot{x}_i^2 > 0. \)

Hereon, the values \( \theta = 1 \) and \( \frac{1}{2} \) in Eqs. (20), (21) and (22) correspond to the time linearization methods referred to as TPET and TPET2, respectively. In the latter, the explicit term \( \frac{d^2 u^n_i}{dx^2} \) in \( Q^n_i \) (cf. Eq. (19)) is approximated as in the TFET2 method.

### 3.4. Quasilinear exponential techniques

These techniques correspond to the iterative solution of (cf. Eq. (19))
\[ \partial t \frac{d^2 u}{dx^2} + \partial J_i u = Q^k_i \equiv -(1 - \theta) \frac{d^2 u^n_i}{dx^2} - (1 - \theta) S^n_i - \partial \theta S^n_i + \partial J_i u^n_i + \frac{u_i - u^n_i}{\Delta t}, \]
\[ x_{i-1} < x < x_{i+1}, \quad t^n < t \leq t^{n+1}, \] (23)
where \( u = u^{k+1} \), until a user’s specified convergence criterion is satisfied (cf. Eq. (9)), and have analogous solutions to those reported in the previous subsection.

Hereon, the values \( \theta = 1 \) and \( \frac{1}{2} \) in Eq. (23) correspond to the quasilinear methods referred to as QPET and QPET2, respectively. In the latter, the explicit term \( \frac{d^2 u^n_i}{dx^2} \) in \( Q^k_i \) is approximated in the same manner as in the TFET2 method, although it could be obtained analytically from either the initial condition or the exponential solutions given by Eqs. (14)–(17) for \( n = 1 \) or \( n > 1 \), respectively. A similar comment applies to the term \( \frac{d^2 u^n_i}{dx^2} \) in \( Q^k_i \).

### 4. Results

In order to assess the accuracy of the exponential time-linearized and quasilinear methods presented in the previous section, we have applied them
to the Nagumo and Fisher equations (cf. Eq. (1)) with $d = 1$ for which $S(u) = xu - \beta u^3$ and $S(u) = \lambda u(1 - u)$, respectively, and compare the results of these methods with those obtained with other (standard) time-linearized and quasi-linear techniques [1,2]. These standard techniques can be written in compact form as

$$u_i^{k+1} - u_i^k = \frac{\Delta t}{h^2} (\theta \delta^2 u_i^{k+1} + (1 - \theta) \delta^2 u_i^k) + \Delta t(\theta S_i^k + \theta J_i^k (u_i^{k+1} - u_i^k) + (1 - \theta) S_i^k),$$

(24)

where $k = n$ for time-linearized methods and a counter for iterative techniques. In the latter case, the same convergence criterion as explained above (cf. Eq. (9)) was used to establish convergence. The cases $h = 1$ and $1/2$ in Eq. (24) correspond to implicit and Crank–Nicolson discretizations which are first- and second-order accurate, respectively, in time. Hereafter, we should refer as the methods of Eq. (24) corresponding to $(k, \theta) = (n, 1/2), (n, 1), (k, 1/2), (k, 1)$ as TL2, TL1, QL2 and QL1, respectively, where T and Q stand for time-linearization and (iterative) quasilinearization, respectively. The exponential time-linearization and exponential quasilinearization methods presented in this paper are also compared with non-standard finite difference schemes [3,4] and nodal integral formulations in this section [8].

4.1. Nagumo’s equation

The Nagumo equation, i.e., Eq. (1) with $S(u) = xu - \beta u^3$, was solved with $d = \alpha = \beta = 1$, $a = 0$, $b = 2$, the following initial condition:

$$u(x, 0) = 1 \quad \text{for} \quad 0 \leq x \leq 1,$$

$$u(x, 0) = 0 \quad \text{for} \quad 1 < x \leq 2,$$

(25)

$u(0, t) = 1$, $u(2, t) = 0$, and some sample results are presented in this section. Fig. 1 (left) shows the numerical solution obtained with TPET, $N = 10$ and $\Delta t = 1$, while Fig. 1 (right) shows the errors incurred by TPET for $\Delta t = 1$ and $N = 10, 50, 100$ and 200, at $t = 1$. These errors are defined as $\text{ERROR} = |u^{NP}(x_i, 1) - u^{401}(x_i, 1)|$ where NP refers to the solution obtained with NP grid points and 401 is the solution obtained with 401 grid points.

The results presented in Fig. 1 (right) indicate that the numerical errors of TPET decrease as the number of grid points is increased. In fact, if the difference between the (unknown) exact solution and the numerical solution obtained with $K$ grid points is $e^K(x_i, t^n) = u_e(x_i, t^n) - u^K(x_i, t^n) = CK^{-p}$ where the subscript $e$ denotes exact solution, $C$ is a constant and $p$ is the convergence order, one can easily deduce that $p = \log_2 |u^K(x_i, t^n) - u^{401}(x_i, t^n)|$ and, for the results presented in Fig. 1 (right), $p = 2$, i.e., TPET is a second-order accurate technique in space.
Fig. 1. Solution of the Nagumo equation obtained with TPET, NP = 11 and Δt = 1 (left) and error (ERROR = |u^{NP}(x_i,1) − u^{401}(x_i,1)|) of the numerical solution (right) at t = 1 for Δt = 1. Right: NP = 11: solid line; NP = 51: dashed line; NP = 101: dashed–dotted line; and NP = 201: dotted line.

Fig. 2. Numerical error (Error(x_i) = |u^M(x_i,1) − u^{QL1}(x_i,1)|) of the solution of the Nagumo equation at t = 1 for NP = 101. M = TPET: solid line; QL2: dashed line; TFET: dashed–dotted line; QFET: dotted line. Top left, top right, bottom left and bottom right correspond to Δt = 0.01, 0.1, 0.001 and 1, respectively.
The results obtained with TPET presented in Fig. 1 (left) are in excellent accord with those obtained by means of the implicit non-standard finite difference method C of Chen et al. [3] for \((\Delta t, h) = (0.001, 0.05), (1, 0.05),\) and, as, the results presented in Fig. 1 (right) indicate for \((\Delta t, h) = (1, 0.2).\) The differences between the results of TPET and the implicit non-standard finite difference method C of Chen et al. [3] were found to be less than \(6 \times 10^{-3}\) for \((\Delta t, h) = (1, 0.2).\)

Figs. 2 and 3 shows the errors of the exponential time-linearization and exponential quasilinearization methods presented in this paper at \(t = 1\) for \(N = 100\) and different time steps, at selected times. The errors in these figures are defined as Error\((x_i) = |u^M(x_i, 1) - u^{QL1}(x_i, 1)|\) where the superscripts \(QL1\) and \(M\) denote the solutions obtained with the \(QL1\) and \(M\) methods, respectively. Fig. 2 indicates that the errors of TPET increase from the boundaries to the middle of the domain and are always less than \(10^{-3}\) for \(\Delta t = 0.01, 0.001\) and 0.1 and less than 0.01 for \(\Delta t = 1.\) The errors of \(QL2\) are large in the middle of the domain for \(\Delta t = 0.01\) and 0.1, and larger than those of TPET for \(\Delta t = 0.001.\) The errors of TFET are of the same order of magnitude as those

Fig. 3. Numerical error \((\text{Error}(x_i) = |u^M(x_i, 1) - u^{QL1}(x_i, 1)|)\) of the solution of the Nagumo equation at \(t = 1\) for \(NP = 101.\) \(M = TL2:\) solid line; \(TL1:\) dashed line; \(TFET2:\) dashed–dotted line. Top left, top right, bottom left and bottom right correspond to \(\Delta t = 0.01, 0.1, 0.001\) and 1, respectively.
of \text{TPET} for $\Delta t = 0.01$, 0.1, and 1, but larger than those of \text{TPET} for $\Delta t = 0.001$, whereas those of \text{QFET} are of the same order as those of \text{TPET} for $\Delta t = 0.001$ and 0.01, but much smaller for $\Delta t = 1$ and 0.1.

The errors exhibited in Fig. 3 indicate that the accuracy of \text{TL2} is comparable to that of \text{TFET2} but lower than that of \text{TL1}.

Figs. 2 and 3 also show that some numerical methods, e.g., \text{QL2}, \text{TL2} and \text{TFET2}, are characterized by large numerical errors at $t = 1$ for $\Delta t = 0.01$ and 0.1, 0.01, 0.1 and 1, and 0.01, 0.1 and 1, respectively. These errors are a consequence of the discontinuity in the initial condition and/or the large time step used in the calculations. These numerical errors decrease as the time step is decreased as shown in the results corresponding to $\Delta t = 0.001$, and indicate that the standard linearization method \text{TL2}, the standard quasilinear technique \text{QL2}, and the exponential time linearization algorithm \text{TFET2} are not as robust as the first-order accurate exponential time-linearization or exponential quasilinearization methods presented in this paper for the Nagumo equation. Moreover, the results presented here and others not shown in the paper indicate that the exponential time-linearization and exponential quasilinearization

![Fig. 4](image-url)
methods corresponding to $\theta = 1$, i.e., TPET and QPET, produce very accurate results even for large time step sizes and grid spacings.

Figs. 4 and 5 illustrate the numerical solutions of the Nagumo equation at $t = 1$ as a function of the number of grid points for $\Delta t = 0.01$, 0.1, 0.001 and 1, and indicate that TL2, QL2 and TPET2 show spikes near $x = 0$ for $\Delta t = 0.1$ and 1, while the TL1, QL1, TFET, TPET and QFET techniques provide monotonic solutions which are nearly independent of the time step for $\Delta t \leq 0.1$. The spikes of the TL2, QL2 and TPET2 methods near $x = 0$ observed for large values of the time step are associated with the discontinuity of the initial condition used in this study and with the diffusion term at the previous time level, i.e., $\frac{C^2 u''}{\Delta t^2}$, cf. Eqs. (18), (23) and (24).

4.2. Fisher’s equation

The Fisher equation, i.e., Eq. (1) with $S(u) = \lambda u(1 - u)$, was solved with $d = \lambda = 1$, $a = -10$, $b = 30$, the following initial condition:
\[ u(x, 0) = 1 \quad \text{for} \quad -10 \leq x \leq 0, \]
\[ u(x, 0) = 0 \quad \text{for} \quad 0 < x \leq 30, \]
\[ u(-10, t) = 1, u(30, t) = 0, \] and some sample results are presented in this section.

Fig. 6 shows the solution of the Fisher equation obtained with the TPET method, NP = 161 and \( \Delta t = 0.125 \) and the numerical errors of several linearization techniques at \( t = 2, 4, 6 \) and 10. This figure indicates that the largest errors of TPET and TPET2 and the solutions obtained with these methods differ by less than \( 6 \times 10^{-3} \) and \( 3 \times 10^{-3} \) from those obtained with TL1 and TL2, respectively, for \( t \leq 10 \). The differences between the results of TPET and TL2 are less than 0.25 and decrease as \( \Delta t \) is decreased; in fact, the largest differences (not shown here) between TPET and TL2 are less than 0.025, for \( \Delta t = 0.0125 \).

Fig. 7 shows the solution of the Fisher equation obtained with the TFET method, NP = 161 and \( \Delta t = 0.125 \) at different times and indicates that the differences between TFET and TL1 are an order of magnitude larger than those

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\textbf{Fig. 6.} Solution of the Fisher equation obtained with TPET and error of the numerical solution (top right and bottom), NP = 101 and \( \Delta t = 0.125 \) (left). Solid line: \( t = 2 \); dashed line: \( t = 4 \); dashed–dotted line: \( t = 6 \); dotted line: \( t = 8 \); +: \( t = 10 \). Top right: error = \( u_{\text{TL1}}(x, t) - u_{\text{TPET}}(x, t) \). Bottom left: error = \( u_{\text{TL2}}(x, t) - u_{\text{TPET}}(x, t) \). Bottom right: error = \( u_{\text{TL2}}(x, t) - u_{\text{TPET2}}(x, t) \).
between TPET and TL1 (cf. Fig. 6); the differences between TFET2 and TL2 are on the same order of magnitude as those between TPET2 and TL2. However, the largest differences between TFET and TL1 and between TFET and TL2 are less than 0.4 for $\Delta t = 0.0125$, but they are one order of magnitude larger than those between TPET and TL1 and between TPET and TL2, respectively, for $\Delta t = 0.0125$. The reason for the deterioration of the accuracy of the TFET technique as the time step is decreased can easily be explained by considering the case that $\Delta t \gg \frac{1}{\omega_1}$ which implies that the contribution of the transient terms to the homogeneous solution of Eq. (13) is very small and, therefore, the complete solution of this equation is affected by transients only through the particular solution. On the other hand, if $\Delta t \ll \frac{1}{\omega_1}$, the contribution of the reaction terms to the homogeneous solution of Eq. (13) is very small and, therefore, the complete solution of this equation is affected by the reaction terms only through the particular solution. This implies that the time step in the TFET technique has to be chosen with care in accord with the characteristic reaction times. As a consequence, the TFET technique is not as robust as the TPET method.

Fig. 7. Solution of the Fisher equation obtained with TFET and error of the numerical solution (top right and bottom), NP = 101 and $\Delta t = 0.125$ (left) and error of the numerical solution (top right and bottom). Solid line: $t = 2$; dashed line: $t = 4$; dashed-dotted line: $t = 6$; dotted line: $t = 8$; +: $t = 10$. Top right: error $= u^{TL1}(x_i, t) - u^{TFET}(x_i, t)$. Bottom left: error $= u^{TL2}(x_i, t) - u^{TFET}(x_i, t)$. Bottom right: error $= u^{TL2}(x_i, t) - u^{TFET2}(x_i, t)$.
Fig. 8 illustrates the solutions obtained with TPET, TL1, and TL2 at selected times as functions of the grid spacing for $\Delta t = 0.125$ and indicates that TPET predicts a faster reaction front for NP = 41 than TL1 which, in turn, predicts a faster front than TL2. Fig. 8 also indicates that TPET, TL1, and TL2 predict the same $u$ profile at different times for NP $\geq$ 161.

The results presented in Fig. 8 are in excellent accord with those of the nodal integral method [8] which predicts that the propagating front speed increases as $h$ is increased for $\Delta t = 0.1$. They are also in excellent accord with those of non-standard finite difference techniques [4]. However, unlike non-standard finite difference methods, the exponential schemes presented in this paper do not require any knowledge of the exact solution of the differential equation and can be easily derived in a rather systematic manner. Compared with nodal integral formulations which require the solution of first- and second-order ordinary differential equations for the space- and time-averaged densities, respectively, the exponential techniques presented in this paper only require the solution of second-order ordinary differential equations.

Fig. 8. Numerical solution of the Fisher equation for $\Delta t = 0.125$. Top left: $t = 2$; top right: 4; bottom left: 6; bottom right: 10: dotted line. NP = 161: solid line = TPET, $x = TL1$, $\bigtriangleup = TL2$; NP = 81: dashed line = TPET, $+$ = TL1, $\triangle = TL2$; NP = 321: dashed-dotted line = TPET, $\bigcirc = TL1$, pentagon = TL2; NP = 641: dotted line = TPET, $*$ = TL1, hexagon = TL2; NP = 41: $\triangledown = TPET$, square = TL1, $\bigtriangledown = TL2$. 
Although not shown here, it was observed that, for NP = 641, \text{TPE}\text{T}_{2}\text{exhibits an oscillation at } x = 0 \text{ at } t = 2, 4, 6 \text{ and } 10 \text{ for } \Delta t = 0.125; \text{ the amplitude of this oscillation is associated with the second-order spatial derivative in the right-and side of Eq. (13), and decreases as the time step is decreased.}

The effects of the time step on the numerical solution obtained with the \text{TPE}\text{T} technique are presented in Fig. 9 which corresponds to NP = 161, and indicates that the \text{TPE}\text{T} method predicts results which are nearly independent of the time step for \( \Delta t \leq 0.0125 \). Although not shown here, for NP = 161, the \text{TPE}\text{T}_{2} method yields front locations and \( u \) profiles which are nearly independent of the time step for \( \Delta t \leq 0.125 \).

5. Conclusions

First- and second-order accurate in time, exponential time-linearization and exponential quasilinearization techniques for one-dimensional reaction–diffusion equation equations have been presented. These techniques are based on the discretization of the time derivative, the freezing of the coefficients in the
resulting linear ordinary differential equations and the piecewise analytical solution of these ordinary differential equations, and yield three-point finite difference expressions that depend in an exponential manner on either the diffusion, reaction and transient terms or the diffusion and reaction terms. It has been found that first-order accurate in time, exponential time-linearization and exponential quasilinearization methods which account for reaction and diffusion processes in the (ordinary) differential operator are more robust than standard second-order accurate finite difference techniques especially for very large time steps and grid sizes; these techniques have also been found more robust than exponential time-linearization and exponential quasilinearization methods that account for reaction, diffusion and transient terms in the differential operator.

For discontinuous initial conditions and large time steps, it has been found that second-order accurate standard and second-order exponential time-linearization and exponential quasilinearization methods predict spikes which are not present in first-order accurate standard and exponential time-linearization and exponential quasilinearization techniques. In addition, it has been found that the time linearization method presented in this paper preserves the fixed points and the linear stability of the diffusionless Nagumo’s equation.

It has also been shown that the first-order accurate (in time) exponential time-linearized and quasilinear methods presented in this paper which account for reaction and diffusion processes in the (ordinary) differential operator are robust with respect to the time step and grid spacing, and provide solutions in excellent accord with those obtained by means of non-standard finite difference schemes and nodal integral formulations for the Nagumo and Fisher equations. However, unlike non-standard finite difference methods, the exponential schemes presented in this paper do not require any knowledge of the exact solution of the differential equation and can be easily derived in a rather systematic manner. Compared with nodal integral formulations which require the solution of first- and second-order ordinary differential equations for the space- and time-averaged densities, the exponential techniques presented here only require the solution of second-order ordinary differential equations.

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References


