Weighted modular inequalities for Hardy–Steklov operators

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Abstract

We characterize weighted modular inequalities of weak and strong type for the Hardy–Steklov operators \( T \) defined by
\[
Tf(x) = g(x) \int_{s(x)}^{h(x)} f(t) \, dt,
\]
where \( g \) is a positive function and \( s, h \) are increasing and continuous functions such that \( s(x) \leq h(x) \) for all \( x \).

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1. Introduction and results

Let \( -\infty < a < b < \infty \) and let \( s, h : (a, b) \to \mathbb{R} \) be increasing and continuous functions such that \( s(x) \leq h(x) \) for all \( x \in (a, b) \). Let \( g \) be a positive function defined on \((a, b)\). Let \( T \) be the Hardy–Steklov operator defined by
\[
Tf(x) = g(x) \int_{s(x)}^{h(x)} f(t) \, dt.
\]

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Particular cases are the Hardy operator \( Tf(x) = \int_0^x f \), the Hardy averaging operators \( Tf(x) = x^\alpha \int_0^x f \), the moving averaging operators \( Tf(x) = \frac{1}{h(x)-s(x)} \int_{s(x)}^{h(x)} f \) and the Steklov operator \( Tf(x) = \int_{s(x)}^{s(x)+1} f \).

Hardy–Steklov operators arise naturally in the theory of delay differential equations and the knowledge of their behaviour may be useful in the study of some Cauchy problems (see [7]).

The weighted strong and weak type \((p,q)\) inequalities for \( T \) have been characterized in [1, 3–5].

In this paper we will characterize weighted modular inequalities of strong and weak type for \( T \), i.e., inequalities of the forms

\[
\Phi_2^{-1} \left( \int_a^b \Phi_2(Tf(x)) u(x) \, dx \right) \leq \Phi_1^{-1} \left( \int_{s(a)}^{h(b)} \Phi_1(Cf(x)) v(x) \, dx \right)
\]

and

\[
\Phi_2^{-1} \left( \Phi_2(\lambda) \int_{\{x \in (a,b): Tf(x) > \lambda\}} u \right) \leq \Phi_1^{-1} \left( \int_{s(a)}^{h(b)} \Phi_1(Cf) v \right),
\]

where \(\Phi_1\) and \(\Phi_2\) are positive, strictly increasing functions defined on \([0, \infty)\) and \( u, v \) are non-negative functions defined on \((a, b)\) and \((s(a), h(b))\), respectively.

The weighted modular inequalities for operators like maximal functions, singular integral, etc., have been extensively studied (see [6] and the references therein). In particular, the weighted modular inequalities for the Hardy operator and the generalized Hardy operators were characterized in [2,8–10].

In the statements and proofs of the results we will need some concepts and properties related to \(N\)-functions. By an \(N\)-function we mean a continuous and convex function \(\Phi\) defined on \([0, \infty)\) such that \(\Phi(s) > 0\) if \(s > 0\), \(\Phi(s)/s \to 0\) when \(s \to 0\) and \(\Phi(s)/s \to \infty\) when \(s \to \infty\). Every \(N\)-function \(\Phi\) admits a representation of the form \(\Phi(x) = \int_0^x \varphi(t) \, dt\), where \(\varphi\) is increasing, continuous by the right at every point and verifies \(\varphi(0) = 0\), \(\varphi(s) > 0\) if \(s > 0\) and \(\varphi(s) \to \infty\) when \(s \to \infty\). The function \(\varphi\) is called the density function of \(\Phi\). Given an \(N\)-function \(\Phi\), the function \(\Psi : [0, \infty) \to \mathbb{R}\) defined by \(\Psi(t) = \sup_{s \geq 0} (st - \Phi(s))\) is also an \(N\)-function called the complementary function of \(\Phi\). Two complementary \(N\)-functions \(\Phi\) and \(\Psi\) verify Young’s inequality: if \(s, t \geq 0\), then \(st \leq \Phi(s) + \Psi(t)\).

Our results are the following ones:

**Theorem 1.** Let \(\Phi_1\) be an \(N\)-function and let \(\Phi_2 : [0, \infty) \to \mathbb{R}\) be a positive strictly increasing continuous function such that \(\Phi_2(0) = 0\) and \(\lim_{t \to \infty} \Phi_2(t) = \infty\). Let us suppose that \(\Phi_1 \circ \Phi_2^{-1}\) is subadditive. Let \(\Psi_1\) be the complementary \(N\)-function of \(\Phi_1\). Let \(u\) and \(v\) be non-negative functions defined on \((a, b)\) and \((s(a), h(b))\), respectively. The following statements are equivalent:

(i) There exists \(C > 0\) such that inequality (1.1) holds for all positive functions \( f \).

(ii) There exists \(C > 0\) such that the inequality

\[
\Phi_2^{-1} \left( \int_{\{x \in (a,b): \int_{s(x)}^{h(x)} f > \lambda\}} \Phi_2(\lambda g) u \right) \leq \Phi_1^{-1} \left( \int_{s(a)}^{h(b)} \Phi_1(Cf(x)) v(x) \, dx \right)
\]

(1.3)
holds for all \( \lambda > 0 \) and all positive functions \( f \).

(iii) There exists \( C > 0 \) such that
\[
\begin{aligned}
\int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\alpha(\lambda, x, y)}{C\lambda v} \right) v \leq \alpha(\lambda, x, y) < \infty
\end{aligned}
\]
holds for all \( \lambda > 0 \) and all \( x, y \in (a, b) \) with \( x < y \) and \( s(y) \leq h(x) \), where
\[
\alpha(\lambda, x, y) = \Phi_1 \circ \Phi_2^{-1} \left( \int_x^y \Phi_2(\lambda g) u \right).
\]

**Theorem 2.** Let \( \Phi_1, \Phi_2, \Psi_1, u \) and \( v \) be as in Theorem 1. Let us suppose that \( g \) is monotone. The following statements are equivalent:

(i) There exists \( C > 0 \) such that inequality (1.2) holds for all \( \lambda > 0 \) and all positive functions \( f \).

(ii) There exists \( C > 0 \) such that
\[
\begin{aligned}
\int_{s(y)}^{h(x)} \Psi_1 \left( \frac{(\inf_{(x,y)} g) \beta(\lambda, x, y)}{C\lambda v} \right) v \leq \beta(\lambda, x, y) < \infty
\end{aligned}
\]
holds for all \( \lambda > 0 \) and all \( x, y \in (a, b) \) with \( x < y \) and \( s(y) \leq h(x) \), where
\[
\beta(\lambda, x, y) = \left( \Phi_1 \circ \Phi_2^{-1} \right) \left( \int_x^y \Phi_2(\lambda) u \right).
\]

Observe that Theorems 1 and 2 include as particular cases the weighted strong and weak type \((p, q)\) inequalities for \( 1 < p \leq q < \infty \). Observe also that if \( g \equiv 1 \), then the strong type inequality (1.1) and the weak type inequality (1.2) are equivalent. However, for general monotone \( g \), (1.1) and (1.2) are not equivalent, even if \( \Phi_1(t) = t^p \) and \( \Phi_2(t) = t^q \), \( 1 < p \leq q < \infty \).

In order to prove the theorems, we will need the following lemma, whose proof can be found in [1]:

**Lemma 1.** Let \( \{ (a_j, b_j) \} \) be the connected components of the open set \( \Omega = \{ x \in (a, b) : s(x) < h(x) \} \). Then

(a) \( (s(a_j), h(b_j)) \cap (s(a_i), h(b_i)) = \emptyset \) for all \( j \neq i \).

(b) For every \( j \) there exists a (finite or infinite) sequence \( \{ m^j_k \} \) of real numbers such that:

(i) \( a_j \leq m^j_k < m^j_{k+1} < b_j \) for all \( k \) and \( j \);

(ii) \( (a_j, b_j) = \bigcup_k (m^j_k, m^j_{k+1}) \) a.e. for all \( j \);

(iii) \( s(m^j_{k+1}) \leq h(m^j_k) \) for all \( j \) and \( k \) and \( s(m^j_{k+1}) = h(m^j_k) \) if \( a_j < m^j_k < m^j_{k+1} < b_j \).

The proof of Theorem 1 is included in Section 2 and the proof of Theorem 2 can be found in Section 3.
2. Proof of Theorem 1

The proof of (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Let $\lambda > 0$, $n \in \mathbb{N}$ and $x, y \in (a, b)$ with $x < y$ and $s(y) \leq h(x)$. If $s(y) = h(x)$, there is nothing to prove. Suppose that $s(y) < h(x)$. Since the function $\Psi_1(t)/t$ increases taking all values from 0 to $\infty$, there exists $\varepsilon > 0$ such that

$$\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\varepsilon}{v + 1/n}\right) \frac{v + 1/n}{\varepsilon} = 2C\lambda, \quad (2.1)$$

where $C$ is the constant of inequality (1.3).

Let $f$ be the function defined by

$$f = \frac{1}{C} \Psi_1\left(\frac{\varepsilon}{v + 1/n}\right) \frac{v + 1/n}{\varepsilon} \chi_{(s(y), h(x))}.$$

If $z \in (x, y)$, we have

$$\int_{s(z)}^{h(z)} \int_{s(y)}^{h(x)} f = \int_{s(y)}^{h(x)} \frac{1}{C} \Psi_1\left(\frac{\varepsilon}{v + 1/n}\right) \frac{v + 1/n}{\varepsilon} = 2\lambda > \lambda.$$

This shows that $(x, y) \subset \{z \in (a, b): \int_{s(z)}^{h(z)} f > \lambda\}$. Then (ii), the inequality $\Phi_1(\frac{\Psi_1(t)}{t}) \leq \Psi_1(t)$ and (2.1) give

$$\alpha(\lambda, x, y) = \Phi_1 \circ \Phi_2^{-1}\left(\int_x^y \Phi_2(\lambda g)u\right)$$

$$\leq \Phi_1 \circ \Phi_2^{-1}\left(\int_{\{z \in (a, b): \int_{s(z)}^{h(z)} f > \lambda\}} \Phi_2(\lambda g)u\right)$$

$$\leq \int_{s(a)}^{h(b)} \Phi_1(C f(t)) \left(v(t) + \frac{1}{n}\right) dt$$

$$= \int_{s(y)}^{h(x)} \Phi_1\left(\Psi_1\left(\frac{\varepsilon}{v + 1/n}\right) \frac{v + 1/n}{\varepsilon}\right) \left(v + \frac{1}{n}\right)$$

$$\leq \int_{s(y)}^{h(x)} \Psi_1\left(\frac{\varepsilon}{v + 1/n}\right) \left(v + \frac{1}{n}\right) = 2C\lambda \varepsilon. \quad (2.2)$$

This inequality ensures $\alpha(\lambda, x, y) < \infty$.

If $\psi_1$ is the density function of $\Psi_1$, it is known that

$$\psi_1(x) \leq x \psi_1(x) \leq \Psi_1(2x). \quad (2.3)$$

On one hand, by (2.2), the right-hand side inequality in (2.3) and (2.1), we have
\[ J = \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\alpha(\lambda, x, y)}{4C\lambda(v + 1/n)} \right) \leq \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\varepsilon}{2(v + 1/n)} \right) \]

\[ \leq 2 \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\varepsilon}{v + 1/n} \right) \frac{v + 1/n}{\varepsilon} = 4C\lambda. \] (2.4)

On the other hand, the left-hand side inequality in (2.3) yields

\[ J \geq \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\alpha(\lambda, x, y)}{4C\lambda(v + 1/n)} \right) \frac{4C\lambda(v + 1/n)}{\alpha(\lambda, x, y)} \]

\[ = \frac{4C\lambda}{\alpha(\lambda, x, y)} \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\alpha(\lambda, x, y)}{4C\lambda(v + 1/n)} \right) \left( v + \frac{1}{n} \right). \] (2.5)

Putting together (2.4) and (2.5), we obtain

\[ \frac{4C\lambda}{\alpha(\lambda, x, y)} \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\alpha(\lambda, x, y)}{4C\lambda(v + 1/n)} \right) \left( v + \frac{1}{n} \right) \leq J \leq 4C\lambda. \]

Letting \( n \to \infty \) and applying the monotone convergence theorem, we get

\[ \int_{s(y)}^{h(x)} \psi_1 \left( \frac{\alpha(\lambda, x, y)}{4C\lambda v} \right) v \leq \alpha(\lambda, x, y). \]

(iii) \( \Rightarrow \) (i). If \( s(x) = h(x) \) for all \( x \in (a, b) \), there is nothing to prove.

Let us suppose that there exists \( z \in (a, b) \) such that \( s(z) < h(z) \). Then \( \Omega = \{ x \in (a, b): s(x) < h(x) \} \) is a nonempty open set. Let \( \{(a_j, b_j)\}_j \) be the collection of the connected components of \( \Omega \) and, for every \( j \), let \( \{m^j_k\} \) be the sequence given by Lemma 1.

For fixed \( j, k \) and \( x \in (m^j_k, m^j_{k+1}) \) we have

\[ Tf(x) = g(x) \int_{s(x)}^{h(x)} f + g(x) \int_{s(m^j_{k+1})}^{h(m^j_k)} f + g(x) \int_{s(m^j_{k+1})}^{h(m^j_k)} f. \]

Then

\[ \int_a^b \Phi_2(Tf)u = \sum_{j,k} \int_{m^j_k}^{m^j_{k+1}} \Phi_2 \left( g(x) \int_{s(x)}^{h(x)} f + g(x) \int_{s(m^j_{k+1})}^{h(m^j_k)} f \right) u(x) \, dx \]

\[ \leq \sum_{j,k} \int_{m^j_k}^{m^j_{k+1}} \Phi_2 \left( g(x) \int_{s(x)}^{3f} u(x) \, dx \right) \]
\[ + \sum_{j,k}^{m_{j+1}^k} \int_{m_{k}^j}^{h(m_{k}^j)} \Phi_2 \left( g(x) \int_{s(m_{k+1}^j)}^{3f} u(x) \, dx \right) \]
\[ + \sum_{j,k}^{m_{j+1}^k} \int_{h(m_{k}^j)}^{h(x)} \Phi_2 \left( g(x) \int_{h(m_{k}^j)}^{3f} u(x) \, dx \right) = (I) + (II) + (III). \]

Let us estimate (III). Let us fix \( j, k \) and consider the sequence \( \{x_n\} \) defined by \( x_0 = m_{j+1}^k + 1 \) and \( \int_{h(x_n)}^{h(x_{n+1})} f = \int_{h(x_n)}^{h(x_{n+1})} f \). This sequence verifies
\[ \int_{h(x_{n+2})}^{h(x_{n+1})} f = \frac{1}{4} \int_{h(m_{k}^j)}^{h(x_n)} f. \]

Let, for every \( n \in \mathbb{N} \), \( f_n = f \chi_{(h(x_{n+2}), h(x_{n+1}))} \). If \( x \in (x_{n+1}, x_n) \) then, by the definition of the sequence \( \{x_n\} \), we have
\[ \int_{h(m_{k}^j)}^{h(x)} 4f_n \geq \int_{h(m_{k}^j)}^{h(x)} 4f_n = \int_{h(m_{k}^j)}^{h(x)} f = \int_{h(m_{k}^j)}^{h(x)} f. \]

This shows that
\[ (x_{n+1}, x_n) \subset E_n = \left\{ x \in (m_{k}^j, m_{k+1}^j) : \int_{h(m_{k}^j)}^{h(x)} 12f_n > \lambda_n \right\}, \quad (2.6) \]

where \( \lambda_n = \int_{h(m_{k}^j)}^{h(x_n)} 3f \).

By the monotonicity of \( \int_{h(m_{k}^j)}^{h(x)} 12f_n \), it is clear that \( E_n \) is an interval of the form \( (\gamma, m_{k+1}^j) \).

Let \( x \in E_n \). Then,
\[ \lambda_n < \int_{h(m_{k}^j)}^{h(x)} 12f_n = \int_{h(m_{k}^j)}^{h(x)} \frac{\alpha(\lambda_n, x, m_{k+1}^j)}{Cv \alpha(\lambda_n, x, m_{k+1}^j)} v \]
\[ \text{or, equivalently,} \]
\[ 2\alpha(\lambda_n, x, m_{k+1}^j) \leq \int_{h(m_{k}^j)}^{h(x)} 24Cf_n \frac{\alpha(\lambda_n, x, m_{k+1}^j)}{\lambda_n Cv} v. \]

Applying Young’s inequality and (iii), we obtain
\[ 2\alpha(\lambda_n, x, m_{k+1}^j) \leq \int_{h(m_{k}^j)}^{h(x)} \Phi_1(24Cf_n)v + \int_{h(m_{k}^j)}^{h(x)} \Psi_1 \left( \frac{\alpha(\lambda_n, x, m_{k+1}^j)}{\lambda_n Cv} \right) v \]

\[ \leq \int_{h(m^j_k)} h(x) \Phi_1(24Cf_n)v + \alpha(\lambda_n, x, m^j_{k+1}), \]

which gives

\[ \alpha(\lambda_n, x, m^j_{k+1}) \leq \int_{h(m^j_k)} h(x) \Phi_1(24Cf_n)v. \]

Since the above inequality holds for all \( x \in E_n \), taking infimum we get

\[ \int_{E_n} \Phi_2(\lambda_n g)u \leq \Phi_2 \circ \Phi_1^{-1} \left( \int_{h(m^j_k)} \Phi_1(24Cf_n)v \right). \] (2.7)

By (2.6), (2.7), the definition of \( f_n \) and the subadditivity of \( \Phi_1 \circ \Phi_2^{-1} \), we conclude

\[
(III) = \sum_{j,k} \int_{m^j_k}^{m^j_{k+1}} \int_{h(m^j_k)}^{h(x)} \Phi_2 \left( g(x) \int_{h(m^j_k)}^{h(x)} 3f \right) u(x) dx
\]

\[
= \sum_{j,k} \sum_{n} x_n \int_{m^j_k}^{m^j_{k+1}} \Phi_2 \left( g(x) \int_{h(m^j_k)}^{h(x)} 3f \right) u(x) dx
\]

\[
\leq \sum_{j,k} \sum_{n} x_n \int_{m^j_k}^{m^j_{k+1}} \Phi_2 \left( g(x) \lambda_n \right) u(x) dx
\]

\[
\leq \sum_{j,k} \sum_{n} \left( \Phi_2 \circ \Phi_1^{-1} \right) \left( \int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf_n)v \right)
\]

\[
= \sum_{j,k} \sum_{n} \left( \Phi_2 \circ \Phi_1^{-1} \right) \left( \int_{h(x_{n+1})}^{h(x_{n+1}+1)} \Phi_1(24Cf)v \right)
\]

\[
\leq \sum_{j,k} \left( \Phi_2 \circ \Phi_1^{-1} \right) \left( \int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf)v \right). \]
The estimation of (I) can be done in a similar way obtaining

\[
(I) \leq \sum_{j,k} \left( \Phi_2 \circ \Phi_1^{-1} \left( \int_{s(m^j_k)} H_{(m^j_{k+1})} \Phi_1(24Cf) v \right) \right).
\]

In order to estimate (II), let \( \lambda_{j,k} = \int_{h(m^j_k)}^{h(m^j_{k+1})} 3f \). By Young’s inequality and (iii) we have

\[
2\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1}) = \int_{s(m^j_{k+1})}^{h(m^j_k)} 6 Cf \frac{\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1})}{C \lambda_{j,k} v} \leq \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6 Cf) v + \int_{s(m^j_{k+1})}^{h(m^j_k)} \Psi_1 \left( \frac{\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1})}{C \lambda_{j,k} v} \right) v.
\]

Therefore

\[
\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1}) \leq \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6 Cf) v,
\]

and this implies

\[
(II) = \sum_{j,k} \int_{m^j_k}^{m^j_{k+1}} \Phi_2 \left( g \int_{s(m^j_{k+1})}^{h(m^j_k)} 3f \right) u \leq \sum_{j,k} \left( \Phi_2 \circ \Phi_1^{-1} \left( \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6 Cf) v \right) \right).
\]

Putting together the estimations of (I)–(III), summing up in \( j \) and \( k \) and applying the subadditivity of \( \Phi_1 \circ \Phi_2^{-1} \), we get (i).

3. Proof of Theorem 2

(i) \( \Rightarrow \) (ii). Let \( \lambda > 0 \) and let \( x, y \in (a, b) \) with \( x < y \) and \( s(y) \leq h(x) \). If \( s(y) = h(x) \), there is nothing to prove. Let us suppose \( s(y) < h(x) \). Let \( \rho \) be a positive number and \( n \in \mathbb{N} \). There exists \( \varepsilon > 0 \) such that

\[
\int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\varepsilon (\inf_{(x,y)} g)}{v + 1/n} \right) v \left( 1 + \frac{1}{n} \right) = (1 + \rho) C \lambda,
\]

where \( C \) is the constant of inequality (1.2).
Let \( f \) be the function defined by
\[
 f = \frac{1}{C} \Psi_1 \left( \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \right) \frac{v + 1/n}{\epsilon(\inf_{(x,y)} g)} \chi_{(s(y),h(x))}.
\]

If \( z \in (x, y) \) we have, by (3.1),
\[
 Tf(z) = g(z) \int_{s(z)} f \geq g(z) \int_{s(y)} f = g(z) \int_{s(y)} \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \cdot \frac{v + 1/n}{\epsilon(\inf_{(x,y)} g)}
\]
\[
 \geq \int_{s(y)} \frac{1}{C} \Psi_1 \left( \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \right) \frac{v + 1/n}{\epsilon} = (1 + \rho) \lambda > \lambda.
\]

We have seen that \( (x, y) \subset \{ z \in (a, b) : Tf(z) > \lambda \} \).

Applying (i), the inequality \( \Phi_1(\Psi_1(t)/t) \leq \Psi_1(t) \) and (3.1), we obtain
\[
 \beta(\lambda, x, y) = (\Phi_1 \circ \Phi_2^{-1})(\Phi_2(\lambda) \int_{x}^{y} u) \leq (\Phi_1 \circ \Phi_2^{-1})(\Phi_2(\lambda) \int_{x}^{y} u)_{\{z \in (a, b) : Tf(z) > \lambda\}}
\]
\[
 \leq \int_{s(a)}^{h(b)} \Phi_1(Cf) v = \int_{s(y)}^{h(x)} \Phi_1 \left( \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \right) \frac{v + 1/n}{\epsilon(\inf_{(x,y)} g)}
\]
\[
 \leq \int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \right) v \leq \int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\epsilon(\inf_{(x,y)} g)}{v + 1/n} \right) \left( v + \frac{1}{n} \right)
\]
\[
 = (1 + \rho) C \lambda \epsilon. \quad (3.2)
\]

The fact that the function \( \Psi_1(t)/t \) increases, together with (3.1) and (3.2) give
\[
 \int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\inf_{(x,y)} g}{(1 + \rho) C \lambda (v + 1/n)} \beta(\lambda, x, y) \right) \frac{v + 1/n}{\beta(\lambda, x, y)}
\]
\[
 = \int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\inf_{(x,y)} g}{(1 + \rho) C \lambda (v + 1/n)} \beta(\lambda, x, y) \right) \left( v + 1/n \right) (1 + \rho) C \lambda \epsilon
\]
\[
 \leq \int_{s(y)}^{h(x)} \Psi_1 \left( \frac{\inf_{(x,y)} g}{v + 1/n} \epsilon \right) \frac{v + 1/n}{\epsilon(1 + \rho) C \lambda} = 1.
\]
Letting \( n \to \infty \) and then \( \rho \to 0 \), we obtain
\[
\frac{h(x)\Psi_{1}\left(\frac{\inf_{(x,y)} g}{C\lambda}\beta(\lambda,x,y)\right)}{\beta(\lambda,x,y)} \leq 1.
\]

(ii) \( \Rightarrow \) (i). If \( \{m^j_k\} \) is the sequence given by Lemma 1,
\[
u\left(\{x \in (a,b) \mid Tf(x) > \lambda\}\right) = \sum_{j,k} \nu\left(\{x \in (m^j_k,m^j_{k+1}) \mid Tf(x) > \lambda\}\right).
\]

For fixed \( k \) and \( j \) we have that if \( x \in (m^j_k,m^j_{k+1}) \), then
\[
Tf(x) = g(x) \int_{s(x)} \int_{s(x)} f + g(x) \int_{s(x)} f + g(x) \int_{s(m^j_{k+1})} f.
\]

By the subadditivity of \( \Phi_{1} \circ \Phi_{2}^{-1} \), it is clear that
\[
\Phi_{1} \circ \Phi_{2}^{-1}\left(\int_{\{x \in (m^j_k,m^j_{k+1}) \mid Tf(x) > \lambda\}} u\right)
\leq \Phi_{1} \circ \Phi_{2}^{-1}\left(\int_{\{x \in (m^j_k,m^j_{k+1}) \mid g(x) \int_{s(x)} f > \lambda/3\}} u\right)
+ \Phi_{1} \circ \Phi_{2}^{-1}\left(\int_{\{x \in (m^j_k,m^j_{k+1}) \mid g(x) \int_{s(m^j_{k+1})} f > \lambda/3\}} u\right)
+ \Phi_{1} \circ \Phi_{2}^{-1}\left(\int_{\{x \in (m^j_k,m^j_{k+1}) \mid g(x) \int_{h(m^j_k)} f > \lambda/3\}} u\right)
= (I) + (II) + (III).
\]

Let us estimate (III). Let \( \{x_n\} \) be the sequence defined as in the proof of Theorem 1. Let
\[
E_n = (x_{n+1}, x_n) \cap \left\{ x \in (m^j_k,m^j_{k+1}) \mid g(x) \int_{h(m^j_k)} f > \lambda/3 \right\}.
\]

If \( x \in E_n \), then
\[
\frac{\lambda}{3} < g(x) \int_{h(m^j_k)} f \leq g(x) \int_{h(m^j_k)} f + 4g(x) \int_{h(x_{n+1})} f.
\]
This implies

\[ \lambda \leq 12 \left( \inf_{E_n} g \right) \int_{h(x_{n+2})} h(x_{n+1}) \]

Let \( \delta_n = \inf E_n \) and \( \gamma_n = \sup E_n \). Since \( g \) is monotone, we can ensure

\[ \lambda \leq 12 \left( \inf_{(\delta_n, \gamma_n)} g \right) \int_{h(x_{n+2})} f. \]

Applying this property and Young’s inequality, we obtain

\[
2\beta(\lambda, \delta_n, \gamma_n) \leq \beta(\lambda, \delta_n, \gamma_n) \frac{24}{\lambda} \left( \inf_{(\delta_n, \gamma_n)} g \right) \int_{h(x_{n+2})} f
\]

\[
= \int_{h(x_{n+2})} 24Cf \frac{\beta(\lambda, \delta_n, \gamma_n) \inf_{(\delta_n, \gamma_n)} g}{C\lambda v} v.
\]

\[
\leq \int_{h(x_{n+2})} \Phi_1(24Cf) v + \int_{h(x_{n+2})} \Psi_1 \left( \frac{\beta(\lambda, \delta_n, \gamma_n) \inf_{(\delta_n, \gamma_n)} g}{C\lambda v} v \right).
\] (3.3)

Since \( s(\gamma_n) \leq s(m_k^{j+1}) \leq h(m_k^j) \), condition (ii) gives

\[
\int_{h(x_{n+2})} \Phi_1(24Cf) v + \int_{h(x_{n+2})} \Psi_1 \left( \frac{\beta(\lambda, \delta_n, \gamma_n) \inf_{(\delta_n, \gamma_n)} g}{C\lambda v} v \right)
\]

\[
\leq \beta(\lambda, \delta_n, \gamma_n).
\]

Taking away this inequality to (3.3), we obtain

\[
\left( \Phi_1 \circ \Phi_2^{-1} \right) \left( \Phi_2(\lambda) \int_{\delta_n} u \right) \leq \int_{h(x_{n+2})} \Phi_1(24Cf) v,
\]

which implies

\[
\left( \Phi_1 \circ \Phi_2^{-1} \right) \left( \frac{\gamma_n}{\delta_n} \int_{E_n} u \right) \leq \int_{h(x_{n+2})} \Phi_1(24Cf) v.
\]

Summing up in \( n \) and applying the subadditivity of \( \Phi_1 \circ \Phi_2^{-1} \), we get

\[ (\text{III}) = \left( \Phi_1 \circ \Phi_2^{-1} \right) \left( \Phi_2(\lambda) \int_{\{ x \in (m_k^j, m_k^{j+1}): g(x) \}^{\int_{h(m_k^j)} f \geq \frac{1}{2}} u \right) \leq \int_{h(x_{n+2})} \Phi_1(24Cf) v. \]
In a similar way, we have

\[
(I) \leq \int_{s(m^j_{k+1})} \Phi_1(24Cf)v.
\]

In order to estimate (II), let

\[
E_{j,k} = \left\{ x \in (m^j_k, m^j_{k+1}) : g(x) \int_{s(m^j_k)}^h f > \frac{\lambda}{3} \right\}.
\]

Since \( g \) is monotone, the set \( E_{j,k} \) is an interval. Working as in the estimation of (III), we prove

\[
(II) = (\Phi_1 \circ \Phi_2^{-1}) \left( \Phi_2(\lambda) \int_{E_{j,k}} u \right) \leq \int_{s(m^j_{k+1})} \Phi_1(24Cf)v.
\]

From the estimations of (I)–(III), we deduce

\[
(\Phi_1 \circ \Phi_2^{-1}) \left( \Phi_2(\lambda) \int_{\{x \in (m^j_k, m^j_{k+1}) : Tf(x) > \lambda\}} u \right) \leq \int_{s(m^j_{k+1})} \Phi_1(24Cf)v.
\]

Summing up in \( k \) and \( j \) and taking into account the subadditivity of \( \Phi_1 \circ \Phi_2^{-1} \), we get

\[
(\Phi_1 \circ \Phi_2^{-1}) \left( \Phi_2(\lambda) \int_{\{x \in (a,b) : Tf(x) > \lambda\}} u \right) \leq \int_{s(a)} \Phi_1(24Cf)v.
\]

References